HOMOLOGY, COHOMOLOGY AND EXTENSIONS OF ORDERED GROUPOIDS

by

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SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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July 2016

Abstract

This thesis contains contributions to the homology, cohomology and extensions of ordered groupoids. We study the simplicial homology of ordered groupoids. We also discuss the (co)homology of the set of identities of ordered groupoids and relate the cohomology of the set of identities of an ordered groupoid to the cohomology of the ordered groupoid. We discuss the β–relation on ordered groupoids; the analogue of the minimum group congruence for inverse semigroups and show that for β–transitive ordered groupoids, the homology of the ordered groupoid is isomorphic to that of its levelled groupoid. In the applications of the discussion on the cohomology of ordered groupoids, we relate the second cohomology group of ordered groupoids to the set of extensions of ordered groupoids with abelian kernel. In particular we show that for an ordered groupoid Q^I obtained from the ordered groupoid Q by attaching the symbol $I \notin Q$ and a Q^I–module $\mathcal{A}^0$ obtained as an extension of the Q–module $\mathcal{A}$, $H^n(Q^I, \mathcal{A}^0)$ is in one-to-one correspondence with the set of extensions of $\mathcal{A}$ by Q. Finally, we follow the approach of Huebschmann but using appropriate constructions for ordered groupoids and verify that our constructions do have the properties required in the arguments of Huebschmann to show that the set of n-fold extensions of an abelian ordered groupoid $\mathcal{A}$ by an ordered groupoid Q is isomorphic to $H^{n+1}(Q^I, \mathcal{A}^0)$.

Acknowledgements

I am greatly thankful to my supervisor Nick Gilbert for being a model, mentor, counsellor and the best support I ever needed throughout my studies at Heriot Watt University. It has been a wonderful experience that I will cherish always. I appreciate his many advice and encouragement which has taken me this far. It is an honour and I take pride in working with him.

My sincere gratitude to Jim Howie for accepting to be my secondary supervisor. I am pleased with the privileges he offered me and encouraging me to participate in gatherings worthwhile to my academic development.

Many thanks to Mark Lawson and Graham Ellis for being part of the examining committee of my dissertation. I am delighted to get in contact with such eminent persons in this area of study in my studies. I give praise to the School of Mathematical Sciences and Heriot Watt at large for the opportunity and support granted me to learn to be a researcher.

Also, thanks to Global Platform Scholarships for paying my fees and providing me with maintenance grant to see me through my studies. I look forward to efficiently contributing to the realisation of their goals.

I am indebted to Bernd Schroers, Brent Everitt and Emmanuel Essel for being part of my academic journey. My journey to this end wouldn't have been without your great contributions.

My appreciation goes to my family and friends who have always been behind me through thick and thin. Kudos to AIMS for being a part of my academic development.

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Chapter 1

Introduction

The concept of an ordered groupoid, a small category in which every morphism is invertible together with a partial order satisfying certain axioms, is a legacy of C. Ehresmann. The theory of ordered groupoids emerged from Ehresmann's work on pseudogroups in the development of differential geometry. In addition to the application in differential geometry from which ordered groupoids emerged, it has been a vital tool in solving problems in areas such as group theory and inverse semigroup theory. P. J. Higgins in [21] gives a good account of the fundamental theory of groupoids and some applications to group theory and topology. In [27], Lawson discusses in detail some connections of the theory of inverse semigroups with the theory of ordered groupoids. One of the vital results in [27] is recorded as the Ehresmann-Schein-Nambooripad theorem which states that the category of inverse semigroups and the category of inductive groupoids are isomorphic. This suggests that one could seek generalisations of ideas in inverse semigroups by interpreting the idea as that of an inductive groupoid and extend it beyond the inductive case to general ordered groupoids. Authors such as Gilbert in [16], Matthews in [33], Lawson in [27], AlYamani, Gilbert and Miller in [2], Steinberg in [41], and AlYamani in [1] have used this connection to solve problems in semigroup theory. We follow the applications and connections between group theory and inverse semigroup theory, and the theory of ordered groupoids to discuss the major results in this thesis.

This thesis is organised into six chapters based upon the main results. In Chapter

1 we revisit some preliminary ideas in the theory of ordered groupoids. We recall the standard definition of ordered groupoids and present some basic terminologies of ordered groupoids and connections with other mathematical objects which are in the scope of the results to be discussed in the following chapters. We commence with a discussion of groupoids following [21]. We explain the concept of ordered groupoids, quotient ordered groupoids and the connection between ordered groupoids and the structures inverse semigroups and left cancellative categories. These ideas are based upon [27], [29], and [1].

In [29], Loganathan presents the homology of an inverse semigroup as the homology of some associated left cancellative category. After discussing the preliminary concepts in line with the core of the results in this thesis, we spend Chapter 2 on discussing the simplicial homology of ordered groupoids with coefficients in a *colouring*. We interpret the simplicial homology of ordered groupoids as the simplicial homology of the associated categories following [29]. The underlying concept follows that of [15] on the simplicial homology of arbitrary small categories. Our approach takes inspiration from that of [14] and [38] and so we stick to some of the terminologies introduced in [14] for consistency but at necessary points we make use of the standard constructions for ordered groupoids. For inductive groupoids, the approach in this chapter can be interpreted as a simplicial homology theory for the associated inverse semigroups.

In Chapter 3 we discuss the (co)homology of the set of identities $E(G)$ of an ordered groupoids and relate the cohomology of $E(G)$ to the cohomology of the ordered groupoid G. We closely follow the approach in [29] with appropriate modifications at some points and provide detailed verifications to concepts which were absent in [29]. In the preliminary discussions we show that for an ordered groupoid G^I obtained by adjoining the symbol $I \notin G$ to the ordered groupoid G and a G^I-module obtained from the G–module A, there are isomorphisms $H^n(\mathcal{L}(G^I), A^0) \cong \mathrm{Ext}^{n-1}_{\mathcal{L}(G)}(KG, A)$ for $n > 0$ where KG is the augmentation module. We use the identification in the preliminary discussions to explain the main result of this chapter which is the construction of the connection between the cohomology of an ordered groupoid and that of its set of identities. The connection presented here is an extension of that shown in [29] for inverse semigroups.

In [29], Loganathan dedicates Section 3 to discussing a relation between the cohomology of an inverse semigroup S and the cohomology of its maximum group homomorphic image. In particular he proves that the classifying space of S is homotopy equivalent to the classifying space of the maximum group homomorphic image of S and the homology of the inverse semigroup reduces to that of its maximum group homomorphic image. Chapter 4 of this thesis is devoted to extending the discussion on the relationship between the cohomology of an inverse semigroup and that of its maximum group homomorphic image by Loganathan to ordered groupoids. Gilbert in [16] introduced *level* groupoids as a tool to investigate the structural properties of ordered groupoids that extends the concept of P–theorem for inverse semigroups by Gomes and Howie in [18]. We commence discussions in this chapter with a review of levelling construction in [16]. We follow with a discussion on the β–relation on ordered groupoids. This is the analogue of the notion of minimum group congruence on inverse semigroups due to Munn in [37]. We discuss the connection between the levelling and β relations on ordered groupoids. Finally we show that for a β–transitive ordered groupoid G, the homology of G is isomorphic to the homology of its level groupoid $G_{\updownarrow}$.

In Chapter 5 we discuss the idea of ordered crossed complexes and ordered chain complexes. These have made appearance in [1] and in the unordered case in [5]. We introduce the concept of five-term exact sequence of low-dimensional homology groups of ordered groupoids, an analogue of the well known five–term exact sequence of low-dimensional homology groups in group theory. We also present the concept of extensions of ordered groupoids and classify the set of extensions of an ordered groupoid with abelian kernel. The classification presented here follows the approach by Gruenberg in [19] for groups. We proceed to show that for an ordered groupoid Q, there is a bijection between the cohomology group $H^2(Q^I, \mathcal{A}^0)$ and the set of equivalence classes of extensions of the abelian ordered groupoid $\mathcal{A}$ by Q. We give an account on how to recover the factor set approach presented by

Matthews in [33] for ordered groupoids. In the case of inductive groupoids, our approach is an alternative proof of the results Lausch in [25] for the associated inverse semigroup.

Finally we spend Chapter 6 in discussing the concept of n-fold extensions of an ordered groupoid $\mathcal{A}$ by Q and show that imposing an appropriate relation on the set of n-fold extensions, the set of equivalence classes is in bijection with the cohomology group $H^{n+1}(Q^I, \mathcal{A}^0)$. The operation on the set of classes is the "Baer sum" of extensions of ordered groupoids. By using this approach we avoid making computations of cocycles in the proof. The result is a generalisation of the relation between the second cohomology group and the set of classes of extensions of ordered groupoids with abelian kernel presented in chapter 5. The approach in this chapter follows that of Huebschmann in [23] on crossed n-fold extensions of groups and cohomology groups.

1.1 Ordered Groupoids

A *groupoid* is a small category in which every morphism is invertible. The set of identities is denoted by G_0. We shall also use $E(G)$ for the set of identities and alternate between the two notations. We write $g \in G(e, f)$ when g is a morphism starting at e and ending f. In particular, $g \in G(e, e)$ is morphism in the vertex group at e. A groupoid G is *connected* if for all $e, f \in G_0$, $G(e, f)$ is non-empty. The composition $g \cdot h$ of morphisms is defined if and only if the source of h is the target of g. Note that we use right composition convention. For brevity we shall write gh for $g \cdot h$. We call a subcategory of G that is also a groupoid a *subgroupoid* of G. Groupoids can be considered as consisting only of morphisms by regarding the objects as identity morphisms at the objects. In this case the morphisms together with composition of morphisms gives an algebraic description of groupoids. The source and target maps are then written as $g\mathbf{d} = gg^{-1}$ and $g\mathbf{r} = g^{-1}g$ respectively. We shall switch between the two notions whenever convenient. The *star* and *costar* of G at an identity e is defined as the sets $\text{star}_G(e) = \{g \in G | gg^{-1} = e\}$ and $\text{costar}_G(e) = \{g \in G | g^{-1}g = e\}$ respectively. A *groupoid–map* $\theta : G \to H$ is just a functor which associates objects and morphisms of G with objects and morphisms of H respectively and preserves composition of morphisms and inverses. We say θ is *star injective* if the restriction $\theta : \mathrm{star}_G(e) \to \mathrm{star}_H(e\theta)$ is injective for all $e \in G_0$. In which case we say θ is an *immersion*. We call a star surjection and star bijection defined analogously a *fibration* and *covering* respectively. Groupoids together with groupoid-maps constitute the category of groupoids denoted by **Gpd**.

Groupoids occur frequently in several situations of study.

Example 1.1.1 An immediate observation is that a group G can be considered as a groupoid with G_0 consisting of the identity element of G.

Example 1.1.2 A set S can be regarded as a trivial groupoid. The elements of the S are the objects of the groupoid together with identity maps as the only morphisms between objects.

Example 1.1.3 Let X be a set which admits a right action by the group Γ. The action groupoid is defined as having morphisms $(x, g) \in X \times \Gamma$ with source $(x, g)\mathbf{d} = x$ and target $(x, g)\mathbf{r} = x \cdot g$. Composition of morphisms is defined by $(x, g)(x \cdot g, h) = (x, gh)$.

Example 1.1.4 The disjoint union of groups is a groupoid.

Definition An *ordered groupoid* is a pair (G, $\leqslant$) where G is a groupoid and $\leqslant$ is a partial order defined on G satisfying the following axioms

OG1 $x \leqslant y \Rightarrow x^{-1} \leqslant y^{-1}$ for all $x, y \in G$,

OG2 let $x, y, u, v \in G$ such that $x \leqslant y$ and $u \leqslant v$. Then $xu \leqslant yv$ whenever the compositions xu and yv exist,

OG3 suppose $x \in G$ and $e \in G_0$ such that $e \leqslant x\mathbf{d}$, then there is a unique element $(x|e)$ called the *restriction* of x to e such that $(x|e)\mathbf{d} = e$ and $(x|e) \leqslant x$,

OG4 if $x \in G$ and $e \in G_0$ such that $e \leqslant x\mathbf{r}$, then there exist a unique element $(e|x)$ called the *corestriction* of x to e such that $(e|x)\mathbf{r} = e$ and $(e|x) \leqslant x$.

For brevity we shall refer to G as an ordered groupoid where necessary. An ordered groupoid is called *inductive* if the pair $(G_0, \leqslant)$ is a meet semilattice. An *ordered functor* $\phi : G \to G'$ of ordered groupoids is an order preserving groupoid–map. That is $g\phi \leqslant h\phi$ whenever $g \leqslant h$. Ordered groupoids together with ordered functors constitute the category of ordered groupoids. We denote by **OGpd** the category of ordered groupoids. We present the following examples to show that ordered groupoids occur in settings.

Example 1.1.5 Every groupoid is an ordered groupoid with ordering defined by

$$x \leqslant y \Leftrightarrow x = y.$$

Example 1.1.6 Let P be a poset and let Ω be a group. Then the product $P \times \Omega$ defines an ordered groupoid with G_0 consisting of the elements of P. The morphisms of G are pairs (p, g) and composition of morphisms defined by $(p, g)(p', g') = (p, gg')$ whenever $p = p'$. We define the ordering on G by $(p, g) \leqslant (p', g')$ if and only if $p \leqslant p'$ and $g = g'$.

Example 1.1.7 Every poset is considered as a trivial ordered groupoid by regarding the elements of the poset as the objects together with identity maps as the only morphisms of the ordered groupoid.

Example 1.1.8 Suppose P is a poset considered as a category with morphisms $p' \to p$ whenever $p \leqslant p'$. A *presheaf* of groups over P is a functor $P \xrightarrow{F} \mathbf{Grp}$ associating every identity $p \in P$ the group F_p. We obtain an ordered groupoid G by the following definitions. Let $G = \bigsqcup_{p \in P} F_p$ the disjoint union of groups hence a groupoid and define the order $\leqslant$ on G by if $x \in F_p$ and $y \in F_{p'}$ then $x \leqslant y$ if an only if $p \leqslant p'$ and $x = (y)\alpha_p^{p'}$ where $\alpha_p^{p'} : F_{p'} \to F_p$ is the induced map by F. Such ordered groupoids are called *Clifford* groupoids. If P is a meet semilattice, then G is an inductive groupoid and its associated inverse semigroup (see section 1.2) is the Clifford inverse semigroup (see details in [27]). Now we present present some ideas which will be used in Chapter 5. The reader is referred to [1] for further details.

Definition A subgroupoid N of an ordered groupoid G is a *normal ordered subgroupoid* if

N01 N has the same set of objects as G: that is N is *wide*,

N02 suppose $n \in N$ and $e \leqslant n\mathbf{d}$ for $e \in G_0$, then the restriction $(n|e)$ of n to e is in N,

N03 if $n \in N$ and $k, h \in G$ such that

1. k and h have an upper bound $g \in G$, that is $k \leqslant g$ and $h \leqslant g$,
2. $h^{-1}nk$ exists in G,

then $h^{-1}nk \in N$.

Remark 1.1.1 [1, Remark 2.3.2] If N is a disjoint union of groups, then the existence of $h^{-1}ak$ implies $h\mathbf{d} = k\mathbf{d}$ and both are restrictions of g and hence by uniqueness of restrictions, they must be equal. This recovers the definition cited in [33]. The following presents the idea of quotients of ordered groupoids by ordered normal subgroupoids. The construction is an extension to ordered groupoids of the method of constructing posets as quotients of preordered sets. A normal ordered subgroupoid N of an ordered groupoid G determines an equivalence relation on G and an ordering which gives rise an ordered groupoid structure on the quotient $G /\!\!/ N$. The following propositions by AlYamani in [1] elaborates these ideas.

Proposition 1.1.1. [1, Lemma 2.3.5] *Suppose N is a normal ordered subgroupoid of the ordered groupoid G. Then the relation $\simeq_N$ defined by*

$$g \simeq_N h \text{ if an only if there exist } a, b, c, d \in N \text{ such that } agb \leqslant h \text{ and } chd \leqslant g$$

is an equivalence relation and the relation

$$[g] \leqslant [h] \text{ if and only if there exist } a, b \in N \text{ such that } agb \leqslant k$$

is a well-defined partial order on the set of equivalence classes $G /\!\!/ N$ of $\simeq_N$.

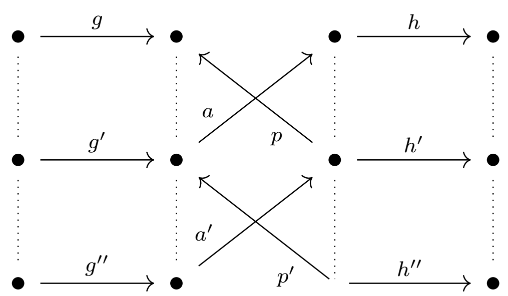
When we restrict the equivalence relation to the set of identities we obtain the following.

Proposition 1.1.2. [1, Lemma 2.3.6] *Suppose G is an ordered groupoid and let $e, f \in G_0$. Then $e \simeq_N f$ if and only if there exist $a, p \in N$ such that*

$$aa^{-1} \leqslant e,\ a^{-1}a = f \text{ and } pp^{-1} \leqslant f,\ p^{-1}p = e\ .$$

The connecting pair (a, p) is called an N-*nexus* between e and f.

Proposition 1.1.3. [1, Lemma 2.3.9] *Suppose $g, h \in G$ such that $g^{-1}g \simeq_N hh^{-1}$ and let (a, p) be an N-nexus between $g^{-1}g$ and hh^{-1}. Define $g' = (aa^{-1}|g)$, $h' = (h|pp^{-1})$, $a' = (pp'|a)$, $p' = (aa^{-1}|p)$, $g'' = (a'a'^{-1}|g)$ and $h'' = (h|p'p'^{-1})$. Diagrammatically we have*



Then $g'ah \simeq_N gp^{-1}h'$ and is independent of the N-nexus chosen.

The proposition above indicates that one could form an equivalence class which we denote by $g \between h = [g'ah] = [gp^{-1}h']$ whenever $g^{-1}g \simeq_N hh^{-1}$. This gives an appropriate candidate for a partial composition operation on equivalent classes. So we state that the composite of the classes $[g]$ and $[h]$ is given as $[g][h] = g \between h$ whenever $g^{-1}g \simeq_N hh^{-1}$. The set of equivalence classes together with the partial order and composition of equivalence classes yields the following.

Proposition 1.1.4. [1, Theorem 2.3.15] *Suppose G is an ordered groupoid and N is a normal ordered subgroupoid of G. Then $G \mathbin{/\!\!/} N$ is an ordered groupoid.*

It is to be noted that are other approaches to treating quotients of ordered groupoids by normal ordered subgroupoids. Lawson in [26] treated quotient ordered groupoids by normal ordered subgroupoids prior to the approach discussed above with inspiration from the work of Joubert in 1966 on congruences on ordered groupoids. We spend the rest of the chapter in explaining the correspondence of the category of ordered groupoids with categories of other mathematical structures in the scope of the results in the following chapters.

1.2 Ordered Groupoids and Inverse Semigroups

This section is devoted to discussing the connection between the category of ordered groupoids and the category of *inverse semigroups*. The inspiration is taken from the from the fact that both categories serve as abstract structures of the *pseudogroup* which was employed by Sophus Lie in the nineteenth century in his generalisation of the Erlanger Program for classifying geometries. We begin with a review of inverse semigroups and follow with a discussion of the connection using the language of category theory. The content of this section is an excerpt of [27].

Definition An *inverse semigroup* is a semigroup S with every element $s \in S$ having a unique element $s^{-1} \in S$ satisfying $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$. The unique element s^{-1} is called the *inverse* of s in S.

Proposition 1.2.1. [27, Proposition 1.4.1] *The elements ss^{-1} and $s^{-1}s$ are idempotents for every $s \in S$ and $(ss^{-1})s = s$ and $s(s^{-1}s) = s$.*

Proof. We observe that

$$\begin{aligned} (s^{-1}s)^2 &= (s^{-1}s)(s^{-1}s) = s^{-1}(ss^{-1}s) = s^{-1}s. \\ (ss^{-1})^2 &= (ss^{-1})(ss^{-1}) = s(s^{-1}ss^{-1}) = ss^{-1}. \end{aligned}$$

It follows from the definition of inverse semigroups that $(ss^{-1})s = s$ and $s(s^{-1}s) = s$. $\square$

Denote the set of idempotents of S by $E(S)$. Set $s\mathbf{d} = ss^{-1}$ and $s\mathbf{r} = s^{-1}s$. Inverse semigroups come equipped with a natural order which we define as follows.

Definition The natural partial order $\leqslant$ on an inverse semigroup S is defined by

$$s \leqslant t \Leftrightarrow s = te \text{ for some } e \in E(S)$$

Proposition 1.2.2. [27, Lemma 1.4.6] *Let S be an inverse semigroup. Then the following are equivalent:*

1. $s \leqslant t$.

2. $s = ft$ *for some* $f \in E(S)$.

3. $s^{-1} \leqslant t^{-1}$.

4. $s = ss^{-1}t$.

5. $s = ts^{-1}s$.

One useful result from the discussion of the natural order is following lemma.

Lemma 1.2.3. [27, Proposition 1.4.8] *Let S be an inverse semigroup.*

1. *If $s \leqslant t$ and $u \leqslant v$ then $su \leqslant tv$.*

2. *$(E(S), \leqslant)$ is a meet semilattice.*

Consider an inverse semigroup S and let $s, t \in S$. The product $s \cdot t$ defined by $s \cdot t = st$ whenever $s^{-1}s = tt^{-1}$ is called the *restricted product*. This is equivalent to saying $s \cdot t$ is defined if $s\mathbf{r} = s\mathbf{d}$. A key result which leads up to that of Theorem 1.2.5 is as follows.

Lemma 1.2.4. [27, Proposition 4.1.1] *Suppose S is an inverse semigroup.*

1. *Let $s \in S$ and $e \in E(S)$ such that $e \leqslant s\boldsymbol{d}$. Then $a = es$ is the unique element in S such that $a \leqslant s$ and $a\boldsymbol{d} = e$.*

2. *Let $s, t \in S$. Then $st = s' \cdot t'$ where $s' = se$, $t' = et$ and $e = s^{-1}stt^{-1}$.*

Theorem 1.2.5. *Let S be S be an inverse semigroup. Then the triple $(S, \cdot, \leqslant)$ is an inductive groupoid.*

We now consider the reverse construction. Suppose G is an ordered groupoid. Let $g, h \in G$ such that $g\mathbf{r}$ and $h\mathbf{d}$ have a greatest lower bound z in G_0. Define the *pseudoproduct* of g and h written $g * h$ by

$$g * h = (z|g)(h|z) \; .$$

This extends the categorical composition in G as incomposable morphisms satisfying the hypothesis will now be composable. The pseudoproduct on inductive groupoids is in fact an everywhere defined binary operation. The following results shows that structure is not only an ordered groupoid but also an inverse semigroup.

Theorem 1.2.6. [27, Proposition 4.1.7] *Suppose G is an inductive groupoid. Then the pair $(G, *)$ is an inverse semigroup.*

Proposition 1.2.7. [27, Proposition 4.1.7]

1. *Suppose G is an inductive groupoid. Then $\boldsymbol{G}(\boldsymbol{S}(G)) = G$.*

2. *Let S be an inverse semigroup. Then $\boldsymbol{S}(\boldsymbol{G}(S)) = S$.*

Proof. 1. Recall that $E(\mathbf{S}(G)) = G_0$. Let $s, t \in \mathbf{S}(G)$ and $e \in G_0$ such that $s = e * t$. Then there exist some $z \in G_0$ such that it is the greatest upper bound of e and $t\mathbf{d}$ by definition. And so $s = e * t = (z|e)(t|z) \leqslant t$ in G. For the converse, suppose $s \leqslant t$ in G. Then $s = (t|s\mathbf{d}) = s\mathbf{d} * t$. Hence the natural order in $\mathbf{S}(G)$ is the order in G. To obtain $\mathbf{G}(\mathbf{S}(G))$, we need to define the restricted product. This is given by $s \cdot t$ is defined whenever $s^{-1} * s = t * t^{-1}$. However, $s^{-1}s = s^{-1} * s$ and $tt^{-1} = t * t^{-1}$ and so the restricted product in $\mathbf{S}(G)$ exist precisely when the product st exist in G. Therefore $\mathbf{G}(\mathbf{S}(G)) = G$ as desired.

2. Suppose $g, h \in \mathbf{G}(S)$ such that z is the greatest lower bound of $g\mathbf{r}$ and $h\mathbf{d}$. Then $g * h = (z|g) \cdot (h|z) = (gz) \cdot (zh) = g \cdot h$ since $z = g^{-1}ghh^{-1}$ by Lemma 1.2.4. Therefore $\mathbf{S}(\mathbf{G}(S)) = S$ as desired.

$\square$

The relationship established suggests that one could use ordered groupoids to investigate structural properties of inverse semigroups. On the other hand, the study of inverse semigroups informs possible analogues for ordered groupoids. Authors such as Gilbert in [16], Matthews in [33], Lawson in [26] and Steinberg in [41] have made contributions in these senses. In this thesis we shall discuss extensions of ideas in inverse semigroup by regarding the ideas as that of an inductive groupoid and hence extend to general ordered groupoids.
Other categorical structures have been shown to be vital tools in studying inverse semigroups. In [29], Loganathan shows that the cohomology of inverse semigroups can be recovered by the cohomology of an associated category which is in fact left cancellative. In the next section we present a functor between the category of ordered groupoids and the category of left cancellative categories. The connection built in the sequel is vital for later discussions in this thesis.

1.3 Ordered Groupoids and Left Cancellative Categories

This section is concerned with the construction of left cancellative categories from ordered groupoids. The construction is inspired by [29]. In [29], Loganathan associates a category $D(S)$ to an inverse semigroup S and later shows that the cohomology of S can be obtained as the cohomology of $D(S)$. Lawson in [28] gives an analogous account for ordered groupoids. He further discusses the connection between left cancellative categories and ordered groupoids. This section is derived from [28] and we leave it to the reader to visit for further details beyond the scope of this chapter. Suppose C is a category and let $a, b, c \in C$ such that the products ab and ac exist. Then C is *left cancellative* if $ab = ac$ implies $b = c$.

Left cancellative categories from ordered groupoids

The following construction originally appeared in [29] for inverse semigroups. The analogue construction for ordered groupoids has appeared in [28], [1] and [33]. Our discussion is adapted from [28].

Let G be an ordered groupoid. We construct a left cancellative category $\mathcal{L}(G)$ from G as follows. The objects of $\mathcal{L}(G)$ are precisely the identities of G, that is $\mathcal{L}(G)_0 = G_0$. The morphisms of $\mathcal{L}(G)$ are defined by

$$\{(e, g) \in G_0 \times G : g\mathbf{d} = gg^{-1} \leqslant e\}$$

with $(e, g)\mathbf{d} = e$ and $(e, g)\mathbf{r} = g^{-1}g = g\mathbf{r}$. It follows that the identity morphisms are of the form (e, e) for some e identity of G. The composition of morphisms is defined by the partial product

$$(e, g)(f, h) = (e, g * h)$$

whenever $f = (e, g)\mathbf{r} = g\mathbf{r}$ else it is undefined. Note that $g * h = (h\mathbf{d}|g)h$. It is easy to check that $\mathcal{L}(G)$ is a category.

Lemma 1.3.1. *Let G be an ordered groupoid. Then $\mathcal{L}(G)$ is a left cancellative category.*

Proof. The goal is to show that the equality $(e, g)(f, h) = (e, g)(f, k)$ of composite morphisms implies $(f, h) = (f, k)$. Take $p \leqslant g\mathbf{d}$ for $p \in G_0$ such that $(h\mathbf{d}|g)\mathbf{d} = p = (k\mathbf{d}|g)\mathbf{d}$.

$$\begin{tikzcd}
\bullet^{e} & & \\
\bullet \arrow[u, dash, dotted] \arrow[r, "g"] & \bullet^{f} & \\
\bullet^{p} \arrow[u, dash, dotted] \arrow[r, "(h\mathbf{d}|g)"] & \bullet \arrow[u, dash, dotted] \arrow[r, "h"] & \bullet
\end{tikzcd}$$

Note that $(h\mathbf{d}|g),\ (k\mathbf{d}|g) \leqslant g$. By the uniqueness of restriction

$$(h\mathbf{d}|g) = (g \mid p) = (k\mathbf{d}|g)\ .$$

Thus left multiplying through $g * h = g * k$ by $(g|p)^{-1}$ we get

$$(h\mathbf{d}|g)^{-1}(h\mathbf{d}|g)h = (k\mathbf{d}|g)^{-1}(k\mathbf{d}|g)k$$

resulting in $h = k$. Therefore $(f, h) = (f, k)$ as desired. $\square$

The construction is functorial hence we obtain a functor

$$\mathcal{L}(-) : \mathbf{OGpd} \to \mathcal{L}C$$

where $\mathcal{L}C$ is the category of left cancellative categories. Suppose $\psi : G \to Q$ is an ordered functor of ordered groupoids. Then $\mathcal{L}(\psi) : \mathcal{L}(G) \to \mathcal{L}(Q)$ is defined by $\mathcal{L}(\psi)(e, g) = (e\psi, g\psi)$. We have that $g\mathbf{d} \leqslant e$ in $\mathcal{L}(G)$ and since ψ is an ordered morphism, $(g\mathbf{d})\psi = (g\psi)\mathbf{d} \leqslant e\psi$ and so $\mathcal{L}(\psi)$ is well defined.

In [28], Lawson constructs a functor from the category of left cancellative categories to the category of ordered groupoids. In particular he shows that each left cancellative category is equivalent to some $\mathcal{L}(G)$ for some ordered groupoid G. He shows that there is a surjection for some ordered groupoids $G \to G(\mathcal{L}(G))$ which is an *ordered embedding* if G is an ordered groupoid with *maximal identities*. We leave the reader to visit [28] for details of the discussion.

Chapter 2

Simplicial Homology of Ordered Groupoids

This chapter is devoted to introducing the concept of *simplicial homology* of ordered groupoids with coefficients in a *colouring.* We adopt the term colouring as it appears in [14] and emphasise that it is exactly a presheaf of modules over ordered groupoids. The idea of simplicial homology of categories has surfaced in many places in literature. We refer the reader to [22], [15], and [34] for comprehensive explanations. In this chapter, we extend the concept of colouring of posets in [14] to ordered groupoids and discuss the idea of simplicial objects in ordered groupoids adapting to the ideas in [15]. We use these preliminary ideas to discuss the idea of simplicial homology of an ordered groupoid with coefficients in a colouring. We begin by discussing the notion of colouring of ordered groupoids.

2.1 Colourings on Ordered Groupoids

In this section we treat the idea of colouring on ordered groupoids. The discussions here are extensions of [14] for posets to ordered groupoids. Consider a unital commutative ring R and denote by Mod_R the category of left R-modules. We define colouring of ordered groupoids as follows.

Definition A *colouring* on an ordered groupoid G is a covariant functor $\mathfrak{F} : \mathcal{L}(G) \to \mathrm{Mod}_R$. The colouring $\mathfrak{F}$ is determined by modules M_e for each $e \in G_0$ and a homomorphism $\zeta_{(e,g)} : M_e \to M_{g^{-1}g}$ for every morphism (e, g) in $\mathcal{L}(G)$. The identity morphism (e, e) at $e \in G_0$ is labelled by the identity homomorphism $\zeta_{(e,e)}$ on M_e and there are module isomorphisms $\zeta_{(gg^{-1},g)} : M_{gg^{-1}} \to M_{g^{-1}g}$ since the action by (gg^{-1}, g) has an inverse action by $(g^{-1}g, g^{-1})$. Suppose $g^{-1}g = f$ then the composition $(e, g)(f, h)$ determines the homomorphism $\zeta_{(e,g*h)} = \zeta_{(e,g)}\zeta_{(f,h)} : M_e \to M_{h^{-1}h}$.

Suppose $\mathfrak{F}$ is a colouring on the ordered groupoid G. Then we shall use the term *coloured* ordered groupoid for the pair $(\mathcal{L}(G), \mathfrak{F})$. Let $(\mathcal{L}(G_1), \mathfrak{F}_1)$ and $(\mathcal{L}(G_2), \mathfrak{F}_2)$ be coloured ordered groupoids. A morphism of coloured ordered groupoids is a pair of maps $(f, \tau) : (\mathcal{L}(G_1), \mathfrak{F}_1) \to (\mathcal{L}(G_2), \mathfrak{F}_2)$ where f is a functor of categories $f : \mathcal{L}(G_1) \to \mathcal{L}(G_2)$ and τ is a natural transformation of functors given by the family of maps $\{\tau_e\}$ for every object e in G_1 defined by $\tau_e : (e)\mathfrak{F}_1 \to ((e)f)\mathfrak{F}_2$ such that the square

$$\begin{tikzcd}
(e)\mathfrak{F}_1 \arrow[r, "\tau_e"] \arrow[d, "\zeta_{(e,g)}"'] & ((e)f)\mathfrak{F}_2 \arrow[d, "\zeta_{((e)f,(g)f)}"] \\
(g^{-1}g)\mathfrak{F}_1 \arrow[r, "\tau_{g^{-1}g}"] & ((g^{-1}g)f)\mathfrak{F}_2
\end{tikzcd}$$

commutes for every morphism $(e, g) \in \mathcal{L}(G_1)$. The natural transformation τ is clearly a morphism of colourings on ordered groupoids. Morphisms of colourings on G such as τ are often called G-maps. One key observation is that the composite functor $f\mathfrak{F}_2$ defines a colouring on G_1. Coloured ordered groupoids together with the morphisms of coloured ordered groupoids constitute a category which we denote by $\mathscr{C}\mathcal{L}G_R$. When we fix the ordered groupoid G, then the collection of colourings on G together with the corresponding G–maps constitutes the category of colourings on G denoted by $\mathrm{Mod}_R^{\mathcal{L}(G)}$. The concept of colouring of groupoids spans over several concepts in algebra and topology.

Example 2.1.1 Let $B \in \mathrm{Mod}_R$. The *constant* colouring $\mathfrak{F}$ on G is given as a covariant functor $\mathfrak{F} : \mathcal{L}(G) \to \mathrm{Mod}_R$ defined on objects by $e \mapsto B$ and the identity map on B for all morphisms in $\mathcal{L}(G)$.

Example 2.1.2 Take the poset P considered as a trivial ordered groupoid. The associated category $\mathcal{L}(P)$ is defined with objects $\mathcal{L}(P)_0 = P_0$ and unique non-invertible order morphisms (e, y) whenever $y \leqslant e$ in P_0. We have morphisms (e, e) as the identity map on every $e \in P_0$. A colouring $\mathfrak{F}$ on P is a covariant functor $\mathfrak{F} : \mathcal{L}(P) \to \text{Mod}_R$ associating every $e \in P_0$ with the module M_e and a module homomorphism $\zeta_{(e,y)} : M_e \to M_y$ for all morphisms (e, y) whenever $y \leqslant e$ in P. The identity morphisms (e, e) at objects in $\mathcal{L}(G)$ gives the identity module homomorphism $\zeta_{(e,e)}$ on M_e and the composition of morphisms in $\mathcal{L}(P)$ gives the module homomorphism composition $\zeta_{(e,y)}\zeta_{(y,z)} = \zeta_{(e,z)} : M_e \to M_z$ whenever $z \leqslant y \leqslant e$ in P. We note that the category $\mathcal{L}(P)$ discussed here is the opposite category of the poset described in [14].

Example 2.1.3 In [24], Khovanov associates some boolean lattices to oriented link diagrams and computes the homology of the associated lattices after labelling the vertices of the lattices with some vector spaces. The labelling of the boolean lattices defines colourings on boolean lattices and hence the Khovanov link invariant obtained by computing the homology of the labelled boolean lattices is the homology of coloured boolean lattices associated with links (see details in [14]).

Product coloured ordered groupoids

Consider the associated categories $\mathcal{L}(G_i)$ of the ordered groupoids G_i for $i = 1, \cdots, n$. By definition of products of categories, we have that $\mathcal{L}(G) = \mathcal{L}(G_1) \times \cdots \times \mathcal{L}(G_n)$ consists of objects n-tuples $(e_1, \cdots, e_n)$ of objects $e_i \in G_i$ $(i = 1, \cdots, n)$ and morphisms from $(e_1, \cdots, e_n)$ to $(e'_1, \cdots, e'_n)$ are defined by $((e_1, g_1), \cdots, (e_n, g_n))$ where $e'_i = g_i^{-1} g_i$ and $(e_i, g_i) \in \mathcal{L}(G_i)(e_i, e'_i)$. Composition of morphisms is carried out component-wise by setting

$$((e_1, g_1), \cdots, (e_n, g_n))((e'_1, g'_1), \cdots, (e'_n, g'_n)) = ((e_1, g_1 * g'_1), \cdots, (e_n, g_n * g'_n))$$

whenever $g_i * g_i'$ is defined for $i = 1, \cdots, n$. The product category $\mathcal{L}(G)$ is equipped with the universal property that for every associated category $\mathcal{L}(H)$ together with projection maps $\pi_i^* : \mathcal{L}(H) \to \mathcal{L}(G_i)$, there exist a unique functor $f : \mathcal{L}(H) \to \mathcal{L}(G)$ making the triangle

$$\begin{tikzcd} & & \mathcal{L}(G_i) \\ \mathcal{L}(G^*) \arrow[rru, bend left, "\pi_i^*"] \arrow[r, "f"] & \mathcal{L}(G) \arrow[ru, "\Pi_i"'] & \end{tikzcd}$$

commute.

Proposition 2.1.1. *Let $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ be the associated categories of the ordered groupoids G_1, G_2. Then there is an equivalence $\mathcal{L}(G_1 \times G_2) \simeq \mathcal{L}(G_1) \times \mathcal{L}(G_2)$ of categories.*

Proof. Consider the direct product $G_1 \times G_2$. The objects and morphisms are defined by pairs (e_1, e_2) and (g_1, g_2) respectively for objects $e_i \in G_i$ $(i = 1, 2)$ and morphims $g_i \in G_i$. The associated left cancellative category, $\mathcal{L}(G_1 \times G_2)$ is hence defined with objects (e_1, e_2) and morphisms $((e_1, e_2), (g_1, g_2))$ for $(g_1, g_2)\mathbf{d} \leqslant (e_1, e_2)$. Since we have a component-wise definition, if $(e_1, e_2) \geqslant (g_1, g_2)\mathbf{d}$ in $G_1 \times G_2$ then $e_1 \geqslant g_1\mathbf{d}$ in G_1 and $e_2 \geqslant g_2\mathbf{d}$ in G_2. So the objects and morphisms of $\mathcal{L}(G)$ has copies of objects and morphism of $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$. Let the identity functors $\mathrm{id} : \mathcal{L}(G_1 \times G_2) \to \mathcal{L}(G_1) \times \mathcal{L}(G_2)$ and $\mathrm{id}^* : \mathcal{L}(G_1) \times \mathcal{L}(G_2) \to \mathcal{L}(G_1 \times G_2)$ so that $\mathrm{id} \circ \mathrm{id}^* = \mathbf{1}_{\mathcal{L}(G_1 \times G_2)}$ and $\mathrm{id}^* \circ \mathrm{id} = \mathbf{1}_{\mathcal{L}(G_1) \times \mathcal{L}(G_2)}$. Therefore $\mathcal{L}(G_1 \times G_2) \simeq \mathcal{L}(G_1) \times \mathcal{L}(G_2)$. $\square$

Suppose G_i for $i = 1, \cdots, n$ are ordered groupoids with colourings $\mathfrak{F}_i$ respectively. Then we define the product coloured ordered groupoid to be the pair $(\mathcal{L}(G), \mathfrak{F})$ where $\mathcal{L}(G)$ denotes the direct product $(\mathcal{L}(G_1) \times \cdots \times \mathcal{L}(G_n)$ and $\mathfrak{F}$ denotes the corresponding collection $\{\mathfrak{F}_i\}_{i=1,\cdots,n}$ of colourings. The colouring $\mathfrak{F}$ labels objects $(e_1, \cdots, e_n)$ with the module $(e_1)\mathfrak{F}_1 \otimes_R \cdots \otimes_R (e_n)\mathfrak{F}_n$ and morphisms $((e_1, g_1), \cdots, (e_n, g_n))$ with module homomorphisms $\zeta_{(e_1,g_1)} \otimes \cdots \otimes \zeta_{(e_n,g_n)} : (e_1, \cdots, e_n)\mathfrak{F} \to (g_1^{-1}g, \cdots g_n^{-1}g_n)\mathfrak{F}$. From the proposition above we get that

$$(\mathcal{L}(G_1), \mathfrak{F}_1) \otimes \cdots \otimes (\mathcal{L}(G_n), \mathfrak{F}_n) \simeq (\mathcal{L}(G_1 \times \cdots \times G_n), \mathfrak{F} = (\{\mathfrak{F}_i\}_{i=1,\cdots,n})) \,.$$

Glued coloured ordered groupoids

It has been earlier described that an ordered functor between ordered groupoids can be used to define new colourings on the ordered groupoids. Here we will discuss some construction of ordered groupoids from ordered morphisms of ordered groupoids and present a colouring which is an extension of the key tool used by Everitt and Turner in [14] in establishing the link between Khovanov's categorification and poset theory.

Let $f : G_1 \to G_2$ be an ordered functor of ordered groupoids. We present the *gluing* of the ordered groupoids along the functor f as follows. The objects of glued ordered groupoid $G_1 \sqcup_f G_2$ are the objects of $G_1 \cup G_2$, the union of the objects of G_1 and G_2. Morphisms of the glued ordered groupoid consist of the union of morphism of G_1 and G_2. Composition of morphisms is defined by the composition of morphisms in G_i otherwise undefined. We define the partial order $\leqslant$ on $G_1 \sqcup_f G_2$ by

- $y \leqslant x$ if $x, y \in G_i$ and $y \leqslant x$ in G_i,
- $y \leqslant x$ if $y \in G_2$, $x \in G_1$ and $(x)f \geqslant y$ in G_2.

We show that $G_1 \sqcup_f G_2$ together with the data above is an ordered groupoid in the following lemma.

Lemma 2.1.2. *$G_1 \sqcup_f G_2$ is an ordered groupoid.*

Proof. It is routine to check that all the axioms of ordered groupoids are satisfied.

1. It is clear that $y \leqslant x$ implies $y^{-1} \leqslant x^{-1}$ for $x, y \in G_i$ by definition. Otherwise for $y \in G_2$ and $x \in G_1$ then $y \leqslant x$ in $G_1 \sqcup_f G_2$ implies $y \leqslant (x)f$. Thus $(x)^{-1}f \geqslant y^{-1}$ and so $(x^{-1})f \geqslant y^{-1}$. Therefore $x^{-1} \geqslant y^{-1}$ in $G_1 \sqcup_f G_2$.

2. If $y \leqslant x$ and $u \leqslant v$ in G_i . It follows from the ordered groupoid structure in G_i that $yu \leqslant xv$ if the compositions yu and xv are defined. On the other hand for $y, u \in G_2$ and $x, v \in G_1$, if $y \leqslant x$ and $u \leqslant v$ via f and the compositions yu and xv are defined. Then the functoriality of f gives $(xv)f = (x)f(v)f \geqslant yu$ in G_2. Hence $yu \leqslant xv$ in $G_2 \sqcup_f G_2$.

3. Suppose e is an object and x is a morphism in G_i. Then by the ordered groupoid structure in G_i, there is a unique restriction of x to e in G_i. Consider e an object of G_2 such that $e \leqslant x\mathbf{d}$ via f for some morphism $x \in G_1$. Then $e \leqslant (x)f\mathbf{d}$ and hence we define the restriction of x to e by the unique element $((x)f|e)$ in G_2. It is evident that $((x)f|e) \leqslant (x)f$ hence $((x)f|e) \leqslant x$ in $G_1 \sqcup_f G_2$ and $((x)f|e)\mathbf{d} = e$. Therefore the restriction of x to e exist in $G_1 \sqcup_f G_2$.

The corestriction follows directly as a consequence of the existence of the restriction and the first axiom. Therefore $G_1 \sqcup_f G_2$ is an ordered groupoid as desired. □

Proposition 2.1.3. *Let f be an ordered functor from the ordered groupoid G_1 to the inductive groupoid G_2. Then the ordered groupoid $G_1 \sqcup_f G_2$ together with the pseudoproduct $*$ is an inverse semigroup.* □

Now we discuss the concept of colourings on glued ordered groupoids. Consider the morphism of coloured ordered groupoids $(f, \tau) : (\mathcal{L}(G_1), \mathfrak{F}_1) \to (\mathcal{L}(G_2), \mathfrak{F}_2)$. We construct the coloured glued ordered groupoid $(\mathcal{L}(G_1 \sqcup_f G_2), \mathfrak{F})$ with $G_1 \sqcup_f G_2$ as the underlying ordered groupoid as follows. The set of objects of $\mathcal{L}(G_1 \sqcup_f G_2)$ consist of the union of the objects of G_1 and G_2. Morphisms of $\mathcal{L}(G_1 \sqcup_f G_2)$ consist of the union of morphisms of $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ and some extra morphisms defined via f. The extra morphisms given as pairs (e, g) are defined whenever e is an object of G_1 and $g \in G_2$ such that $gg^{-1} \leqslant e$ via f.

A colouring on the glued ordered groupoid is defined as a covariant functor $\mathfrak{F} : \mathcal{L}(G_1 \sqcup_f G_2) \to \mathrm{Mod}_R$ given on objects by $(e)\mathfrak{F} = (e)\mathfrak{F}_i$ for $e \in G_i$ and on morphism by the assignments

- $\zeta_{(e_i,g_i)} : (e_i)\mathfrak{F}_i \to (g_i^{-1}g_i)\mathfrak{F}_i$ for morphisms $(e_i, g_i) \in G_i$,

- suppose $e \in G_1$ and $g \in G_2$ such that $gg^{-1} \leqslant (e)f$. Then we note that $(e)f$ is key to defining the morphism (e,g) defined via f and the composition $f\mathfrak{F}_2$ defines a colouring on G_1. Hence we can define the natural transformation $\tau_e : (e)\mathfrak{F}_1 \to ((e)f)\mathfrak{F}_2$. We obtain the following diagram of module homomorphisms

$$\begin{tikzcd} (e)\mathfrak{F}_1 \arrow[r, "\tau_e"] & ((e)f)\mathfrak{F}_2 \arrow[d, "\zeta_{((e)f,gg^{-1})}"] & \\ & (gg^{-1})\mathfrak{F}_2 \arrow[r, "\zeta_{(gg^{-1},g)}"] & (g^{-1}g)\mathfrak{F}_2 \end{tikzcd}$$

Hence we set $\tau_e \circ \zeta_{((e)f,g)} : (e)\mathfrak{F}_1 \to (g^{-1}g)\mathfrak{F}_2$ as the label for the morphism (e,g) following that $\zeta_{((e)f,g)} = \zeta_{((e)f,gg^{-1})}\zeta_{(gg^{-1},g)}$.

Lemma 2.1.4. *$(\mathcal{L}(G_1 \sqcup_f G_2), \mathfrak{F})$ is a coloured ordered groupoid.*

Proof. The goal of the proof is to show the functoriality of $\mathfrak{F}$. We consider the following situations and note that other situations are direct consequences of these.

1. Let $(e,g),(m,h)$ be morphisms in G_i. Then if the composition $(e,g)(m,h)$ is defined in G_i, we have $(e,g)(m,h) = (e, g*h)$ and since $\mathfrak{F}_i$ is a colouring, it follows that $((e,g)(m,h))\mathfrak{F}_i = (e,g)\mathfrak{F}_i(m,h)\mathfrak{F}_i$.

2. Suppose e is an object of G_1, g is a morphism in G_2 with $(e)f \geqslant gg^{-1}$. Hence (e,g) is an arrow in $\mathcal{L}(G_1 \sqcup_f G_2)$. Let (m,h) be a morphism in $\mathcal{L}(G_2)$. Suppose the composition $(e,g)(m,h)$ is defined in $\mathcal{L}(G_1 \sqcup_f G_2)$, then we have $(e,g)(m,h) = (e, g*h) = (e, (h\mathbf{d}|g)h)$. Applying $\mathfrak{F}$ gives

$$\begin{tikzcd} (e)\mathfrak{F}_1 \arrow[r, "\tau_e"] & ((e)f)\mathfrak{F}_2 \arrow[d, "\zeta_{((e)f,gg^{-1})}"] & & \\ & (gg^{-1})\mathfrak{F}_2 \arrow[r, "\zeta_{(gg^{-1},g)}"] \arrow[d, "\zeta_{(gg^{-1},(h\mathbf{d}|g)\mathbf{d})}"] & (g^{-1}g)\mathfrak{F}_2 \arrow[d, "\zeta_{(g^{-1}g,hh^{-1})}"] & \\ & (h\mathbf{d}|g)\mathbf{d})\mathfrak{F}_2 \arrow[r, "\zeta_{(h\mathbf{d}|g)\mathbf{d},(h\mathbf{d}|g))}"] & (hh^{-1})\mathfrak{F}_2 \arrow[r, "\zeta_{(hh^{-1},h)}"] & (h^{-1}h)\mathfrak{F}_2 \end{tikzcd}$$

and hence

$$\begin{aligned} ((e,g)(m,h))\mathfrak{F} &= (e,(h\mathbf{d}|g)h)\mathfrak{F} \\ &= \tau_e \circ \zeta_{((e)f,(h\mathbf{d}|g)\mathbf{d})}\zeta_{((h\mathbf{d}|g)\mathbf{d},(h\mathbf{d}|g)h)} = \tau_e \circ \zeta_{((e)f,(h\mathbf{d}|g)h)} \end{aligned}$$

and

$$\begin{aligned}(e,g)\mathfrak{F}(m,h)\mathfrak{F} &= (e,g)\mathfrak{F}(m,h)\mathfrak{F}_2 \\ &= \tau_e \circ \zeta_{((e)f,gg^{-1})}\zeta_{(gg^{-1},g)}\zeta_{(m,h)} = \tau_e \circ \zeta_{((e)f,(h\mathbf{d}|g)h)}\ .\end{aligned}$$

Thus from the functoriality of $\mathfrak{F}_2$ and naturality of τ we have $((e,g)(m,h))\mathfrak{F} = (e,g)\mathfrak{F}(m,h)\mathfrak{F}$. Therefore $(\mathcal{L}(G_1 \sqcup_f G_2), \mathfrak{F})$ is a coloured ordered groupoid. □

2.2 Simplicial sets

Simplicial sets offer an algebraic analogue of using triangulation techniques for studying topological spaces. This approach is a widely known concept and can be found in the literature including [34], [20], [43], [12] and [15]. The main goal of the following paragraph is to discuss the concept of *nerve* and simplicial objects in a category and hence the categories $\mathcal{L}(G)$ associated with ordered groupoids. Details of the sequel are contained in [43] and [15]. We proceed with the formal definition of simplicial sets.

Definition A *simplicial set* K is a collection of sets K_n, for $n \in \mathbb{N}$ together with maps $\partial_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ for $0 \leqslant i \leqslant n$ such that

- $\partial_i\partial_j = \partial_{j-1}\partial_i$ for $i < j$,
- $s_i s_j = s_{j+1} s_i$ for $i \leqslant j$,
- $\partial_i s_j = s_{j-1}\partial_i$ for $i < j$,
  $\partial_j s_j = id_{K_n} = \partial_{j+1} s_j$,
  $\partial_i s_j = s_j \partial_{i-1}$ for $i > j+1$.

The elements of K_n for any n are called *n–simplices*. The maps ∂_i and s_i are called *face* and *degeneracy* maps respectively. A simplicial map is a family of degree zero maps $\tau_n : K_n \to L_n$ which is compatible with the face and degeneracy maps. That is $\tau_n\partial_i = \partial_i\tau_{n-1}$ and $\tau_n s_i = s_i\tau_{n+1}$. The collection of simplicial sets together with simplicial maps constitute the category of simplicial sets. Our discussion so far presents simplices in the category of sets. It is noted that the discussion can be carried out in other categories and so we describe the concept of a simplicial structure in an arbitrary mathematical structure and hence ordered groupoids.

Definition The *simplicial* category Δ consist of the collection of finite ordered sets $[n] = \{0 < 1 \cdots < n\}$ where $n \in \mathbb{N}$ as objects and morphisms nondecreasing monotonic functions. More precisely if $\mu \in \mathrm{Hom}([n],[m])$ and $i < j$ then $(i)\mu \leqslant (j)\mu$.

Define the degeneracy and face maps in Δ by

$$(j)s_i = \left\{ \begin{array}{ll} j & \text{if } j \leqslant i \\ j+1 & \text{if } j \geqslant i \end{array} \right\} \qquad (j)\partial_i = \left\{ \begin{array}{ll} j & \text{if } j \leqslant i \\ j-1 & \text{if } j \geqslant i \end{array} \right\}$$

Every morphism of Δ can be described as a composition of the face and degeneracy maps.

Definition Let $\mathcal{C}$ be a category. A *simplicial object* in $\mathcal{C}$ is a contravariant functor $F : \Delta \to \mathcal{C}$, alternatively, a covariant functor $\Delta^{op} \xrightarrow{F} \mathcal{C}$. The elements of $([n])F$ are called n-simplices. Thus we rephrase our earlier discussion by referring to simplicial sets as a simplicial object in the category of sets. Whenever we fix the category $\mathcal{C}$ we obtain the corresponding name of the simplicial object accordingly. For example the names simplicial groups, simplicial ordered groupoids, simplicial Lie Algebras and simplicial R–modules denotes simplicial objects in the category of groups, ordered groupids, Lie Algebra and R–modules respectively.

Example 2.2.4 Let $\mathcal{C}$ be a category. A widely known example of a simplicial set is the *nerve* $N\mathcal{C}$ of $\mathcal{C}$ presented as follows. $N\mathcal{C}$ is the collection of sets $N\mathcal{C}_n$ consist of sequences of n-composable morphisms in $\mathcal{C}$ for $n > 0$ and $\mathcal{C}_0$ for $n = 0$. The face and degeneracy maps of $N\mathcal{C}$ are defined by

$$\begin{aligned}
(x_1 \cdots x_n)\partial_i &= \begin{cases} x_2 \cdots x_n & \text{if } i = 0 \\ x_1 \cdots (x_i \circ x_{i+1}) \cdots x_n & \text{if } 0 < i < n \\ x_1 \cdots x_{n-1} & \text{if } i = n \end{cases} \\
(x_1 \cdots x_n)s_i &= \quad x_1 \cdots x_i x_i \cdots x_n \quad \text{for all } i.
\end{aligned}$$

Proposition 2.2.1. *The nerve of the Cartesian product category $\mathcal{A} \times \mathcal{B}$ is given by $N\mathcal{A} \times N\mathcal{B}$ with face and degeneracy maps $\partial_i = (\partial_i^{\mathcal{A}}, \partial_i^{\mathcal{B}})$ and $s_i = (s_i^{\mathcal{A}}, s_i^{\mathcal{B}})$ respectively.*

Simplicial objects together with the corresponding natural transformations constitute the functor category $\mathcal{C}^{\Delta}$. A covariant functor $\mathbf{F} : \mathcal{C} \to \mathcal{A}$ induces a functor $\mathcal{C}^{\Delta} \to \mathcal{A}^{\Delta}$ defined by $F \mapsto F\mathbf{F}$. One could easily think of the example of the free abelian group functor $\mathbf{Sets} \to \mathbf{Ab}$. Suppose $S \in \mathbf{Sets}$, then $(S)\mathbf{F}$ is the free abelian group generated by S. The induced map sends a simplicial set to a simplicial abelian group.

In line with the interest of our study, we consider the colouring functor $\mathfrak{F} : \mathcal{L}(G) \to \mathrm{Mod}_R$. The colouring induces a functor $\mathcal{L}(G)^{\Delta} \to \mathrm{Mod}_R^{\Delta}$. We present a simplicial object $S(\mathcal{L}(G), \mathfrak{F})$ in Mod_R defined via $\mathfrak{F}$ with motivation from [15] as follows. Let $N\mathcal{L}(G)$ be the nerve of $\mathcal{L}(G)$ and set $((e_1, g_1) \cdots (e_n, g_n))\mathfrak{F} = (e_1)\mathfrak{F}$. For brevity we use the notation $\alpha_i = (e_i, g_i)$ and denote the n-simplex $\alpha_1 \cdots \alpha_n$ by α. We define the simplicial object $S(\mathcal{L}(G), \mathfrak{F})$ by

$$S_n(\mathcal{L}(G), \mathfrak{F}) = \begin{cases} 0 & n < 0 \\ \bigoplus_{e \in N\mathcal{L}(G)_0} (e)\mathfrak{F} & n = 0 \\ \bigoplus_{\alpha \in N\mathcal{L}(G)_n} (e_1)\mathfrak{F} & n > 0 \end{cases}$$

The face and degeneracy maps are defined by

$$(\lambda\cdot[\alpha_1\cdots\alpha_n])\tilde{\partial}_i=\left\{\begin{array}{ll}(\alpha\partial_0)\mathfrak{F}, & i=0\\ (\alpha\partial_i)\mathfrak{F}, & 0<i\leqslant n\end{array}\right\}\qquad(\lambda\cdot[\alpha_1\cdots\alpha_n])\tilde{s}_i=\left\{\begin{array}{l}(\alpha s_i)\mathfrak{F}\end{array}\right\}$$

where $\lambda\in(e_1)\mathfrak{F}$ and $\lambda\cdot[\alpha_1\cdots\alpha_n]$ is an n-simplex in $S(\mathcal{L}(G),\mathfrak{F})$ indexed by the n-simplex $\alpha_1\cdots\alpha_n$ in $N\mathcal{L}(G)$.

2.3 Chain Complexes

In the following paragraphs we present the construction of chain complexes from the simplicial object in the category $\mathcal{L}(G)$ discussed in the previous section. This is a consequence of the correspondence between simplicial abelian groups and chain complexes described by Dold and Kan independently in 1957 (see [43], [15] for details).

We construct a chain complex $\mathbb{S}(\mathcal{L}(G),\mathfrak{F})$ from the simplicial object $S(\mathcal{L}(G),\mathfrak{F})$ as follows definitions. We set $\mathbb{S}_n(\mathcal{L}(G),\mathfrak{F})=S_n(\mathcal{L}(G),\mathfrak{F})$. The differential $d_n:\mathbb{S}_n(\mathcal{L}(G),\mathfrak{F})\to\mathbb{S}_{n-1}(\mathcal{L}(G),\mathfrak{F})$ is induced by the face map of the simplicial object $S(\mathcal{L}(G),\mathfrak{F})$. It is explicitly given as $d=\sum_{i=0}^{i=n}(-1)^i\tilde{\partial}_i$ and so for a typical computation we have

$$\begin{aligned}(\lambda\cdot[\alpha_1\cdots\alpha_n])d_n &= (\lambda\triangleleft\alpha_1)\cdot[\alpha_2\cdots\alpha_n]+\sum_{i=1}^{n-1}(-1)^i\lambda\cdot[\alpha_1\cdots\widehat{\alpha_i\alpha_{i+1}}\cdots\alpha_n]\\ &+ (-1)^n\lambda\cdot[\alpha_1\cdots\alpha_{n-1}]\end{aligned}$$

where $\lambda\triangleleft\alpha_1$ denotes the action of α_1 on λ and $\widehat{\alpha_i\alpha_{i+1}}$ is the composition of morphisms in the category $\mathcal{L}(G)$ defined via the pseudoproduct on G. That is $\widehat{\alpha_i\alpha_{i+1}}=(e_i,g_i)*(e_{i+1},g_{i+1})=(e_i,(g_{i+1}\mathbf{d}|g_i)g_{i+1})$. The results of the differential on the right hand side of the equation denotes components of $(\alpha)\mathfrak{F}$ indexed by the $(n-1)$-simplices of $N\mathcal{L}(G)$ regarded as elements of $\mathbb{S}_{n-1}(\mathcal{L}(G),\mathfrak{F})$. The following lemma follows from the Dold-Kan correspondence between simplicial abelian groups and chain complexes (seem[43]).

Lemma 2.3.1. *$\mathbb{S}_*(\mathcal{L}(G), \mathfrak{F})$ together with the differential d is a chain complex.*

Proof. The aim of the proof is to show that $d^2 = 0$, that is $d_n d_{n-1} = 0$. Thus we first compute the differential d_n followed by d_{n-1} by our convention. Following the definition of the differential, the terms of $(\lambda \cdot [\alpha_1 \cdots \alpha_n])d_n$ have the forms $(\lambda \triangleleft \alpha_1) \cdot [\alpha_2 \cdots \alpha_n]$, $\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i \alpha_{i+1}} \cdots \alpha_n]$ and $\lambda \cdot [\alpha_1 \cdots \alpha_{n-1}]$. Applying the second differential on the first term gives us

$$(\lambda \triangleleft \alpha_1\alpha_2)\cdot[\alpha_3 \cdots \alpha_n] + \sum_{i=2}^{n-1}(-1)^i(\lambda \triangleleft \alpha_1)\cdot[\alpha_2 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n] + (-1)^n(\lambda \triangleleft \alpha_1)\cdot[\alpha_2 \cdots \alpha_{n-1}]$$

Also we get

$$\begin{aligned} & (\lambda \triangleleft \alpha_1)\cdot[\alpha_2 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n] + \sum_{j=1}^{i-1}(-1)^j \lambda \cdot [\alpha_1 \cdots \widehat{\alpha_j\alpha_{j+1}} \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n] \\ + & \sum_{k=i+1}^{n-1}(-1)^k \lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \widehat{\alpha_k\alpha_{k+1}} \cdots \alpha_n] + (-1)^n \lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_{n-1}] \end{aligned}$$

from computing d_{n-1} of the terms of the form $\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n]$. Finally, applying d_{n-1} to the terms of the form $\lambda \cdot [\alpha_1 \cdots \alpha_{n-1}]$ yields

$$(\lambda \triangleleft \alpha_1)\cdot[\alpha_2 \cdots \alpha_{n-1}] + \sum_{i=1}^{n-2}(-1)^{i-1}\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_{n-1}] + (-1)^{n-1}\lambda \cdot [\alpha_1 \cdots \alpha_{n-2}]$$

We now point out that all the terms after applying the second differential occur in pairs with opposite signs hence cancel out in the sum in $d_n d_{n-1}$ to give the desired result.

The terms $\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_r\alpha_{r+1}} \cdots \widehat{\alpha_s\alpha_{s+1}} \cdots \alpha_n]$ occurs exactly twice arising from the computations $(\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_r\alpha_{r+1}} \cdots \alpha_n])d_{n-1}$ and $(\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_s\alpha_{s+1}} \cdots \alpha_n])d_{n-1}$ with opposite signs $(-1)^{r+s-1}$ and $(-1)^{r+s}$ respectively. Thus the two terms cancel out in pairs in the summation in $d_n d_{n-1}$.

The pair of the term of the form $(\lambda \triangleleft \alpha_1\alpha_2)\cdot[\alpha_3 \cdots \alpha_n]$ obtained from computing $((\lambda \triangleleft \alpha_1)\cdot[\alpha_2 \cdots \alpha_n])\, d_{n-1}$ arises from computing $\left(\lambda \cdot [\widehat{\alpha_1\alpha_2} \cdots \alpha_n]\right) d_{n-1}$. The pair have signs 1 and -1 respectively hence cancel out in the summation in $d_n d_{n-1}$.

The differentials $(\lambda \cdot [\alpha_1 \cdots \alpha_{n-1}])\, d_{n-1}$ and $\left(\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n]\right) d_{n-1}$ yield terms of the form $\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i \alpha_{i+1}} \cdots \alpha_{n-1}]$ with signs $(-1)^{n+i}$ and $(-1)^{n+i-1}$ respectively. Thus all such pairs have opposite signs and so cancel out in $d_n d_{n-1}$.

Similarly the terms of the form $(\lambda \lhd \alpha_1) \cdot [\alpha_1 \cdots \widehat{\alpha_i \alpha_{i+1}} \cdots \alpha_n]$ arises in pairs from computing $((\lambda \lhd \alpha_1) \cdot [\alpha_1 \cdots \alpha_n])\, d_{n-1}$ and $\big(\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i \alpha_{i+1}} \cdots \alpha_n]\big)\, d_{n-1}$. The computations yield the signs $(-1)^{i-1}$ and $(-1)^i$ which are opposite hence the pairs cancel out in summation in $d_n d_{n-1}$.

Also the differentials $((\lambda \lhd \alpha_1) \cdot [\alpha_2 \cdots \alpha_n])\, d_{n-1}$ and $(\lambda \cdot [\alpha_1 \cdots \alpha_{n-1}])\, d_{n-1}$ both yield the terms of the form $(\lambda \lhd \alpha_1) \cdot [\alpha_2 \cdots \alpha_{n-1}]$ however with opposite signs $(-1)^{n-1}$ and $(-1)^n$ respectively. Therefore all such terms cancel out in sum in $d_n d_{n-1}$.

Finally the term $\lambda \cdot [\alpha_1 \cdots \alpha_{n-2}]$ from $(\lambda \cdot [\alpha_1 \cdots \alpha_{-1}])d_{n-1}$ with the sign $(-1)^{2n-1}$ arises in pair. Its pair is obtained from $(\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_{n-1}\alpha_n}])d_{n-1}$ with the sign $(-1)^{2n-2}$. Thus in summation all such pairs cancel out.

Therefore all the terms in $d_n d_{n-1}$ arise in pairs with opposite signs and cancel out in sum leaving $d^2 = d_n d_{n-1} = 0$ as desired. $\square$

Remark 2.3.1 A morphism of the categories $\mathcal{L}(G_1) \xrightarrow{f} \mathcal{L}(G_2)$ induces

1. a simplicial map $N\mathcal{L}(G_1) \to N\mathcal{L}(G_2)$ defined on simplices by
$$\alpha_1 \cdots \alpha_n \mapsto (\alpha_1)f \cdots (\alpha_n)f \ ,$$
2. a morphism of categories of colourings $\mathrm{Mod}_R^{\mathcal{L}(G_2)} \xrightarrow{f_*} \mathrm{Mod}_R^{\mathcal{L}(G_1)}$ defined by $\mathfrak{F}_2 \mapsto f\mathfrak{F}_2$. We assign a natural transformation $\Gamma : \mathfrak{F}_2 \to \mathfrak{F}_2'$ with the natural transformation $f_*\Gamma : f_*\mathfrak{F}_2 \to f_*\mathfrak{F}_2'$ given as $f_*\Gamma_e = \Gamma_{f(e)} : ((e)f)\mathfrak{F}_2 \to ((e)f)\mathfrak{F}_2'$,
3. a map of simplicial complexes $\mathbb{S}_*(\mathcal{L}(G_2), \mathfrak{F}_2) \xrightarrow{f_*} \mathbb{S}_*(\mathcal{L}(G_1), f\mathfrak{F}_2)$ via the *pull-back*, defined for an n–simplex α in $N\mathcal{L}(G_1)$ and $\lambda \in (e_1)\mathfrak{F}_2$ by
$$(\lambda \cdot [\alpha])f_* = \lambda \cdot [\alpha f] \ ,$$
4. a group homomorphism $f^* : \mathbb{S}_*(\mathcal{L}(G_1), f_*\mathfrak{F}_2) \to \mathbb{S}_*(\mathcal{L}(G_2), \mathfrak{F}_2)$ via the *push-*

forward defined by

$$(\lambda \cdot [\beta])f^* = \sum_{[\alpha]\in([\beta])f^{-1}} \lambda \cdot [\alpha]$$

where f^{-1} is finite, for $\lambda \in ((e_1)f)\mathfrak{F}_2$ and β a simplex in $N\mathcal{L}(G_2)$. We set $(\lambda \cdot [\beta])f^* = 0$ if f^{-1} is empty. The induced homomorphism is not necessarily a chain map as it is evident from the case when f is an immersion and not a fibration.

Lemma 2.3.2. *Let $\mathcal{L}(G_2) \xrightarrow{f} \mathcal{L}(G_3)$ be a functor of categories. Then the induced map $\mathbb{S}_*(\mathcal{L}(G_3), \mathfrak{F}_3) \xrightarrow{f_*} \mathbb{S}_*(\mathcal{L}(G_2), f\mathfrak{F}_3)$ is a chain map. Suppose we have a functor $\mathcal{L}(G_1) \xrightarrow{g} \mathcal{L}(G_2)$. Then $(gf)_* = f_*g_* : \mathbb{S}_*(\mathcal{L}(G_3), \mathfrak{F}_3) \xrightarrow{g_*} \mathbb{S}_*(\mathcal{L}(G_1), (gf)_*\mathfrak{F}_3)$.*

Morphisms of colourings on ordered groupoids induces some maps. Consider the natural transformation $\tau : \mathfrak{F} \to \mathfrak{F}'$ of colourings on G. Then the pair $(\mathrm{id}, \tau) : (\mathcal{L}(G), \mathfrak{F}) \to (\mathcal{L}(G), \mathfrak{F}')$ is a morphism of coloured ordered groupoids. The natural transformation induces a map $\tau^* : \mathbb{S}_n(\mathcal{L}(G), \mathfrak{F}) \to \mathbb{S}_n(\mathcal{L}(G), \mathfrak{F}')$ defined by

$$(\lambda \cdot [\alpha_1 \cdots \alpha_n])\tau^* = (\lambda \triangleleft \tau_{e_1}) \cdot [\alpha_1 \cdots \alpha_n]$$

for $\lambda \in (e_1)\mathfrak{F}$ and $\alpha_1 \cdots \alpha_n \in N\mathcal{L}(G)$. Let $\alpha = \alpha_1 \cdots \alpha_n$. We notice that $((\lambda \cdot [\alpha])\tau^*)\, d_n$ gives

$$(\lambda \triangleleft \tau_{e_1}) \cdot [\alpha_2 \cdots \alpha_n] + \sum_{i=1}^{n-1}(-1)^i(\lambda \triangleleft \tau_{e_1}) \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n] + (\lambda \triangleleft \tau_{e_1}) \cdot [\alpha_1 \cdots \alpha_{n-1}]$$

and $((\lambda \cdot [\alpha])d_n)\, \tau^*$ gives

$$\tau^*(\lambda \triangleleft \alpha_1) \cdot [\alpha_2 \cdots \alpha_n] + \sum_{i=1}^{n-1}(-1)^i\tau^*\lambda \cdot [\alpha_1 \cdots \widehat{\alpha_i\alpha_{i+1}} \cdots \alpha_n] + \tau^*(\lambda \triangleleft \tau_{e_1}) \cdot [\alpha_1 \cdots \alpha_{n-1}]$$

thus $((\lambda \cdot [\alpha])\tau^*)\, d_n$ and $((\lambda \cdot [\alpha])d_n)\, \tau^*$ are equal. Therefore τ^* is a chain map. So we are led to the following lemma.

Lemma 2.3.3. *Let $\tau : \mathfrak{F}_3 \to \mathfrak{F}_3'$ be a morphism of colourings on the ordered groupoid G_3 and suppose $\mathcal{L}(G_1) \xrightarrow{f} \mathcal{L}(G_2) \xrightarrow{g} \mathcal{L}(G_3)$ is a sequence of functors. Then the* *diagram*

$$\begin{tikzcd}
\mathbb{S}_*(\mathcal{L}(G_3),\mathfrak{F}_3) \arrow[r, "f_*"] \arrow[d, "\tau^*"] & \mathbb{S}_*(\mathcal{L}(G_2), f_*\mathfrak{F}_3) \arrow[r, "(gf)_*"] \arrow[d, "f_*\tau^*"] & \mathbb{S}_*(\mathcal{L}(G_1),(gf)\mathfrak{F}_3) \arrow[d, "(gf)_*\tau^*"] \\
\mathbb{S}_*(\mathcal{L}(G_3),\mathfrak{F}_3') \arrow[r, "f_*"] & \mathbb{S}_*(\mathcal{L}(G_2), f\mathfrak{F}_3') \arrow[r, "(gf)_*"] & \mathbb{S}_*(\mathcal{L}(G_1),(gf)\mathfrak{F}_3')
\end{tikzcd}$$

commute.

2.4 Simplicial homology of ordered groupoids

The construction of the chain complexes in the previous section defines a functor,

$$\mathbb{S} : \mathscr{C}\mathcal{L}G_R \to \mathrm{Ch}_R$$

from the category of coloured ordered groupoids to the category of chain complexes over R–modules. In this section we present the concept of simplicial homology of ordered groupoids with coefficients in a colouring. We proceed with the following formal definitions.

The *homology* H of the chain complex $\mathbb{S}(\mathcal{L}(G),\mathfrak{F})$ is defined as

$$H_n\left(\mathbb{S}(\mathcal{L}(G),\mathfrak{F})\right) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}\ .$$

So we define the *simplicial homology* of the ordered groupoid G with coefficient in a colouring $\mathfrak{F}$ by

$$H_n\left(\mathcal{L}(G),\mathfrak{F}\right) = H_n\left(\mathbb{S}(\mathcal{L}(G),\mathfrak{F})\right)\ .$$

Let $\mathfrak{F}$ be the trivial colouring which associates each $e \in G_0$ with $B \in \mathrm{Mod}_{\mathcal{L}(G)}$ and $\mathrm{id}_B = \alpha_i$ (that is $\mathcal{L}(G)$ acts trivially on B). The colouring $\mathfrak{F}$ is the constant system of coefficient for simplicial chain complexes. In passing from an n-simplex to an $n-1$-simplex, there is no coefficient shifts as $\lambda \triangleleft \alpha_1 = \lambda$ where $\lambda \in \mathfrak{F}(e_1)$. Here the coefficients do not depend on $e \in G_0$. The constant $\mathbb{Z}$ coefficient system appears as the coefficient system in the traditional homology theory of ordered groupoids.

Chapter 3

(Co)homology of the set of Identities of Ordered Groupoids

This chapter is concerned with discussing some (co)homological results for ordered groupoids. The relationship between the cohomology groups of an ordered groupoid G and the cohomology groups of its set of identities $E(G)$ is the paramount goal here. A closely related idea is found in [29]. We shall make use of an alternative description of (co)homology of ordered groupoids with coefficient in some functor discussed in Chapter 2 using Ext and Tor groups. The (co)homology theories developed here are the same as the simplicial homology theories in the previous Chapter (see [15], [45] for detail). In the treatment here, a significant tool is the concept of *module* over ordered groupoids. Loganathan in [29] discusses the notion of *adjoint* module $\mathbb{Z}S$ over inverse semigroups and further presents a relation between the cohomology of an inverse semigroup with an adjoined identity $H^n(\mathcal{L}(S^I), \mathcal{A}^0)$ and the cohomology of the *augmentation* ideal $H^{n-1}(KS, \mathcal{A})$ for $n \geqslant 1$ and $\mathcal{A}$ an $\mathcal{L}(S)$–module. He uses these preliminary results to discuss the relationship between the cohomology of an inverse semigroup and that of its semilattice of idempotents. As an application to this result, he shows that for a free inverse semigroup S, $H^n(\mathcal{L}(S), \mathcal{A}) \cong H^n(E(S), \mathcal{A})$ for $n \geqslant 2$ and $\mathcal{A}$ an $\mathcal{L}(S)$–module.

The main result of this chapter is presented in Theorem 3.4.5, and is an analogue of Proposition 4.4 of [29]. This chapter is organised into five sections. The first section commences with a discussion of the concept of modules over ordered groupoids. We present the idea of adjoint module $\mathbb{Z}G$ and hence augmentation module KG over ordered groupoids following [6] for unordered groupoids. In section two we present some functors on the module categories $\mathrm{Mod}_{\mathcal{L}(G)}$ and $\mathrm{Mod}_{E(G)}$. In particular we show that the functors are adjoint pair. We dedicate the third section to discussing the (co)homology of ordered groupoids and that of its set of identities. We show that there are isomorphisms $H^n(E(G), \mathcal{A}) \cong \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$ for $\mathcal{A}$ a G–module. In the fourth section we present the cohomology of ordered groupoids with an adjoined identity. One of the key results discussed in this section is the identification $H^n(\mathcal{L}(G^I), \mathcal{A}^0) \cong \mathrm{Ext}^{n-1}_{\mathcal{L}(G)}(KG, \mathcal{A})$. Finally section five uses the arguments in the previous sections to establish a relationship between the cohomology groups of ordered groupoids and that of their set of identities.

3.1 Modules over ordered groupoids

This section presents the concept of modules over ordered groupoids following [29]. The idea of modules over unordered groupoids can be found in [5]. The reference [33] contains a discussion of the idea of modules over ordered groupoids. In [1], AlYamani presents a comprehensive discussion of modules over ordered groupoids which can be easily seen as an extension of [17], where Gilbert discusses the concept of modules over inverse semigroups with motivation from [29].

Let G be an ordered groupoid. Denote the set of identities of G by $E(G)$. The set $E(G)$ is a partially ordered set with respect to the restriction of the partial order on G. We recall that this poset can itself be regarded as an ordered groupoid. This is often called a *trivial* groupoid. As an ordered groupoid, the only morphisms between objects of $E(G)$ are the identity maps. The poset $E(G)$ itself is naturally a category with a unique non invertible and non-identity morphism $e \to f$ whenever $e > f$. Now the associated category $\mathcal{L}(E(G))$ of the trivial groupoid $E(G)$ has objects the elements of $E(G)$ and morphisms the identity maps and non invertible order maps between distinct comparable elements. Hence the associated category $\mathcal{L}(E(G))$ and the poset $E(G)$ regarded as a category are identical. Thus we shall use the category $E(G)$ in the sequel and agree that every result holds for the category $\mathcal{L}(E(G))$. We define modules over ordered groupoids and infer the corresponding definition for the ordered subgroupoids $E(G)$. Our argument is adapted from [1], [29] and [6].

Definition Let G be an ordered groupoid. A module $\mathcal{A}$ over G is a functor $\mathcal{A}$: $\mathcal{L}(G) \to \mathbf{Ab}$ from $\mathcal{L}(G)$ into the category of abelian groups defined by the following data,

- $\mathcal{A}$ determines a family of abelian groups $\{\mathcal{A}_e\}_{e\in E(G)}$,
- each arrow $(gg^{-1}, g) \in \mathcal{L}(G)$ gives a group isomorphism $\lhd(gg^{-1}, g) : \mathcal{A}_{gg^{-1}} \to \mathcal{A}_{g^{-1}g}$ such that
  1. $\lhd g : \mathcal{A}_e \to \mathcal{A}_e$ is the identity map on $\mathcal{A}_e$ whenever g is the identity at e,
  2. suppose the composition gh exist in G, then $\lhd g \cdot \lhd h = \lhd(gh) : \mathcal{A}_{gg^{-1}} \to \mathcal{A}_{h^{-1}h}$,
- every order morphism (e, f) gives a homomorphism $\alpha^e_f : \mathcal{A}_e \to \mathcal{A}_f$ such that $\alpha^e_e : \mathcal{A}_e \to \mathcal{A}_e$ is the identity on and $\alpha^e_f \alpha^f_z = \alpha^e_z$ whenever $z \leqslant f \leqslant e$ in $E(G)$,
- if $e \leqslant gg^{-1}$ so that $(g|e)$ is the restriction of g to e finishing at n, then the diagram

$$\begin{tikzcd}
\mathcal{A}_{gg^{-1}} \arrow[r, "\lhd g"] \arrow[d, "\alpha^{gg^{-1}}_e"'] & \mathcal{A}_{g^{-1}g} \arrow[d, "\alpha^{g^{-1}g}_n"] \\
\mathcal{A}_e \arrow[r, "\lhd(g|e)"'] & \mathcal{A}_n
\end{tikzcd}$$

  commutes.

Morphisms of G–modules are called G–maps. A G-map is precisely a natural transformation of functors. We denote by $\mathrm{Mod}_{\mathcal{L}(G)}$ the functor category consisting of G-modules and it corresponding natural transformations. Let $\mathcal{A}$ and $\mathcal{A}'$ be G-modules and $\nu : \mathcal{A} \to \mathcal{A}'$ a morphism of G-modules. Then diagrams of the form

$$\begin{tikzcd} \mathcal{A}_e \arrow[r, "\nu_e"] \arrow[d, "\alpha^e_f"'] & \mathcal{A}'_e \arrow[d, "\alpha^e_f"] \\ \mathcal{A}_f \arrow[r, "\nu_f"'] & \mathcal{A}'_f \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_m \arrow[r, "\nu_m"] \arrow[d, "\triangleleft g"'] & \mathcal{A}'_m \arrow[d, "\triangleleft g"] \\ \mathcal{A}_n \arrow[r, "\nu_n"'] & \mathcal{A}'_n \end{tikzcd}$$

commute in $\mathrm{Mod}_{\mathcal{L}(G)}$ for $e \geqslant f$ in $E(G)$ and $g \in G(m, n)$ respectively. The definition of an $E(G)$–module follows from the above definition as a functor $E(G) \to \mathbf{Ab}$ satisfying the above conditions. $E(G)$–modules together with the corresponding $E(G)$–maps constitute the category of modules over $E(G)$ denoted by $\mathrm{Mod}_{E(G)}$. Now we present some G–modules which play vital roles in the discussion of the main result of this chapter.

3.1.1 Adjoint modules over ordered groupoids

This section is concerned with the description of *adjoint modules* over ordered groupoids. The case of unordered groupoids can be found in [5]. The construction presented here is parallel to that for the group ring for groups. Our discussion takes its inspiration from [29].

Let $\mathcal{L}(G)$ be the associated category of the ordered groupoid G. We define the functor $R(G) : \mathcal{L}(G) \to \mathbf{Sets}$ from $\mathcal{L}(G)$ into the category of sets as follows. The set valued functor associates each object $e \in \mathcal{L}(G)$ with the set defined by $R(G)_e = \{g \in G : g^{-1}g = e\}$. That is the set $R(G)_e$ consist of morphisms of G with target e and so $R(G)_e$ is precisely the costar of G at e. A morphism $h \in G(e, f)$ gives a map $R(G)_e \to R(G)_f$ given by $g \mapsto gh$ for $g \in R(G)_e$ and $gh \in R(G)_f$. Suppose $h' \in G(f, z)$ then the product hh' gives a map $R(G)_e \to R(G)_z$ defined by $g \mapsto ghh'$ where $g \in R(G)_e$ and $ghh' \in R(G)_z$. Let $e, f \in E(G)$ with $e \geqslant f$. Then $g \in R(G)_e$ has a corestriction $(f|g)$ with target f and hence $(f|g) \in R(G)_f$. This gives a map $R(G)_e \to R(G)_f$.

Definition Let $\mathcal{L}(G)$ be the associated category of the ordered groupoid G. The *adjoint module* $\mathbb{Z}G$ over G is the functor $\mathbb{Z}G : \mathcal{L}(G) \to \mathbf{Ab}$ from $\mathcal{L}(G)$ to the category of abelian groups defined as follows. It is the composite of the set–valued functor R and the free abelian group functor **Sets** $\to$ **Ab**, that is

$$\mathbb{Z}G : \mathcal{L}(G) \to \mathbf{Sets} \to \mathbf{Ab}\ .$$

$\mathbb{Z}G$ associates every $e \in G_0$ with the free abelian group $(\mathbb{Z}G)_e$ on $R(G)_e$. So $(\mathbb{Z}G)_e$ can be written as the formal sum $\sum_{g\in R(G)_e} n_g g$ with $n_g \in \mathbb{Z}$. Let g be a basis element in $(\mathbb{Z}G)_e$ and (e,h) be a morphism in $\mathcal{L}(G)$. Then the action of (e,h) on g written $g \triangleleft (e,h)$ is the composite of the actions of (e,hh^{-1}) and (hh^{-1},h). We explain the (e,hh^{-1}) and (hh^{-1},h) actions as follows. We note that $e \geqslant hh^{-1} \in E(G)$ hence there is a unique morphism $(hh^{-1}|g) \in (\mathbb{Z}G)_{hh^{-1}}$, the corestriction of g to hh^{-1}. The morphism $(hh^{-1}|g)$ is a basis element in $(\mathbb{Z}G)_{hh^{-1}}$. Hence the action of (e,hh^{-1}) on g is given by

$$g \triangleleft (e,hh^{-1}) = (hh^{-1}|g) \in (\mathbb{Z}G)_{hh^{-1}}\ .$$

So the action gives a homomorphism $(\mathbb{Z}G)_e \xrightarrow{\triangleleft(e,hh^{-1})} (\mathbb{Z}G)_{hh^{-1}}$. Now the G–action is given as follows. Let $\sum_{\bar{g}\in R(G)_{hh^{-1}}} n_{\bar{g}}\bar{g} \in (\mathbb{Z}G)_{hh^{-1}}$, then we define the action of (hh^{-1},h) on the formal sum by

$$\sum_{\bar{g}\in R(G)_{hh^{-1}}} n_{\bar{g}}\bar{g} \triangleleft h = \sum_{\bar{g}\in R(G)_{hh^{-1}}} n_{\bar{g}}(\bar{g}h) \in (\mathbb{Z}G)_{h^{-1}h}\ .$$

Thus the action of (hh^{-1},h) gives a group homomorphism $(\mathbb{Z}G)_{hh^{-1}} \xrightarrow{\triangleleft(hh^{-1},h)} (\mathbb{Z}G)_{h^{-1}h}$. So we obtain a homomorphism

$$(\mathbb{Z}G)_e \xrightarrow{\triangleleft(e,hh^{-1})} (\mathbb{Z}G)_{hh^{-1}} \xrightarrow{\triangleleft(hh^{-1},h)} (\mathbb{Z}G)_{h^{-1}h}$$

for each morphism (e,h) in $\mathcal{L}(G)$. The action of the identity morphism at e gives the homomorphism $\triangleleft(e,e) : (\mathbb{Z}G)_e \to (\mathbb{Z}G)_e$ which is the identity map on $(\mathbb{Z}G)_e$. Suppose (e,h) and (f,h') are composable morphisms in $\mathcal{L}(G)$. Then we obtain a homomorphism $\triangleleft(e,h)\cdot\triangleleft(f,h') = \triangleleft((e,h)(f,h')) : (\mathbb{Z}G)_e \to (\mathbb{Z}G)_{h'^{-1}h'}$.

Suppose $\theta : G \to G'$ is an ordered morphism of ordered groupoids. Then there is an induced map $\mathbb{Z}G \to \mathbb{Z}G'$ and so $\mathbb{Z}$ defines a functor

$$\mathbb{Z}(-) : \mathbf{OGpd} \to \mathrm{Mod}_{\mathcal{L}(G)}$$

from the category of ordered groupoids to the category of modules over ordered groupoids. A less interesting but vital element of the category $\mathrm{Mod}_{\mathcal{L}(G)}$ is the constant (diagonal) module denoted by $\mathbb{Z}$ for $\mathbb{Z} \in \mathbf{Ab}$. We denote by $\Delta : \mathbf{Ab} \to \mathrm{Mod}_{\mathcal{L}(G)}$ the constant functor which associates each abelian group $\mathbb{Z} \in \mathbf{Ab}$ the constant G-module $\Delta\mathbb{Z}$ giving by $(\Delta\mathbb{Z})_e = \mathbb{Z}$ and the identity map on $\mathbb{Z}$ for morphisms in $\mathcal{L}(G)$. We shall use $\Delta\mathbb{Z}$ for the constant module $\mathbb{Z}$ in the sequel to avoid possible confusion with functor $\mathbb{Z}(-)$.

Definition Let G be an ordered groupoid. The epimorphism $\mathbb{Z}G \xrightarrow{\varepsilon} \Delta\mathbb{Z}$ defined by $\sum\limits_{g \in R(G)_e} n_g g \mapsto \sum\limits_{g \in R(G)_e} n_g$ is called the *augmentation map*. The kernel of ε is called the *augmentation* module over G which we shall denote by KG. We recall that any ordered morphism $\theta : G \to G'$ of ordered groupoids induces a module–map $\mathbb{Z}G \to \mathbb{Z}G'$ hence $KG \to KG'$. Thus the augmentation module defines a functor $K(-) : \mathbf{OGpd} \to \mathrm{Mod}_{\mathcal{L}(G)}$.

Lemma 3.1.1. *Suppose KG is augmentation module over the ordered groupoid G. Then a $\mathbb{Z}$–basis of the free abelian group $(KG)_e$ consist of all $g - 1_e$ with g a non-identity in $R(G)_e$.*

Proof. Let $a = \sum\limits_{g \in R(G)_e} n_g g$ be a typical element in $(KG)_e$. Then $\sum\limits_{g \in R(G)_e} n_g = 0$ implies

$$a = \sum_{g \in R(G)_e} n_g g = \sum_{g \in R(G)_e} n_g g - \sum_{g \in R(G)_e} n_g = \sum_{g \in R(G)_e} n_g (g - 1_e) \ .$$

□

3.2 Correspondence between functors on module categories

In this section we shall present some functors on the module category of ordered groupoids and that of their ordered subgroupoid $E(G)$, and show that they are indeed adjoint pairs. The reference [29] is the source of motivation for this section. Loganathan in [29] shows that for an inverse semigroup S, the restriction $\mathrm{Mod}_{\mathcal{L}(S)} \to \mathrm{Mod}_{E(S)}$ has an adjoint pair. Our main result is stated in Theorem 3.2.4. In the case of inductive groupoids, our discussion supplies details omitted in [29] in the analogous discussions by Loganathan for the associated inverse semigroups.

3.2.1 $E(G)$–modules from G–modules.

In this section we present a functor from modules over ordered groupoids to modules over their set of identities. That is a functor $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{E(G)}$.

Let G be an ordered groupoid with set of identities $E(G)$. Then the poset of identities $E(G)$ is a ordered subgroupoid of the ordered groupoid G. The inclusion $E(G) \hookrightarrow G$ gives the inclusion $E(G) = \mathcal{L}(E(G)) \hookrightarrow \mathcal{L}(G)$ and the composition with a G–module defines a functor $E(G) \to \mathbf{Ab}$ which is in fact an $E(G)$–module. Also any G–map generates an $E(G)$–map via the composition with the inclusion.

Hence the inclusion induces a covariant functor

$$\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{E(G)} \ .$$

The induced functor is called the *restriction along the inclusion*. In particular consider $\mathbb{Z}G \in \mathrm{Mod}_{\mathcal{L}(G)}$, we write $\mathbb{Z}G|_{E(G)}$ for the $E(G)$–module obtained by the restriction of $\mathbb{Z}G$ along the inclusion. Suppose $\mathcal{A}$ is an $E(G)$–module and let $\phi : \mathcal{A} \to \mathbb{Z}G|_{\mathrm{Mod}_{E(G)}}$ be an $E(G)$-map. Then ϕ is the family of maps $\phi_e : \mathcal{A}_e \to (\mathbb{Z}G)_e$ for all objects $e \in E(G)$. Suppose that e and f are comparable objects in $E(G)$, say $e \geqslant f$. Then the unique order map $e \to f$ in $E(G)$ defines a unique action on $a \in \mathcal{A}_e$ expressed via the $E(G)$-map ϕ as

$a \triangleleft (e, f) = a \triangleleft (\alpha^e_f \cdot \phi_f) = a \triangleleft (\phi_e \cdot \beta^e_f)$ where $\alpha^e_f : \mathcal{A}_e \to \mathcal{A}_f$ and $\beta^e_f : (\mathbb{Z}G)_e \to (\mathbb{Z}G)_f$ are the group homomorphisms induced by the order morphism (e, f) in $\mathcal{L}(G)$ on $\mathcal{A}$ and $\mathbb{Z}G$ respectively. This is depicted via the commutative diagram

$$\begin{tikzcd}
\mathcal{A}_e \arrow[r, "\phi_e"] \arrow[d, "\gamma^e_f"'] & (\mathbb{Z}G)_e \arrow[d, "\beta^e_f"] \\
\mathcal{A}_f \arrow[r, "\phi_f"] & (\mathbb{Z}G)_f
\end{tikzcd}$$

The restriction functor is exact as it is evident that is does not modify the underlying sets (costar) and so preserve injections and epimorphisms. We will show that the restriction along the inclusion admits a left adjoint.

3.2.2 G-modules from $E(G)$–modules.

In [29], Loganathan shows that for an inverse semigroup S, the restriction functor $\mathrm{Mod}_{\mathcal{L}(S)} \to \mathrm{Mod}_{E(S)}$ admits a left adjoint. In the following paragraphs we construct a functor

$$\mathrm{Mod}_{E(G)} \to \mathrm{Mod}_{\mathcal{L}(G)}$$

for an ordered groupoid G, which admits a left and right composition with the restriction functor induced by the inclusion of $E(G)$ into $\mathcal{L}(G)$ to give the identity $\mathrm{Mod}_{\mathcal{L}(G)}$–functor and $\mathrm{Mod}_{E(G)}$–functor respectively. Our arguments are adapted from [29] but with appropriate modification to align with the theory of ordered groupoids.

Let G be an ordered groupoid and suppose that $\mathcal{A}$ is an $E(G)$–module. Define the covariant functor $\mathscr{H} : \mathrm{Mod}_{E(G)} \to \mathrm{Mod}_{\mathcal{L}(G)}$ by

$$(\mathscr{H}\mathcal{A})_e = \bigoplus_{g \in R(G)_e} \mathcal{A}_{gg^{-1}} \; .$$

We have that $(\mathscr{H}\mathcal{A})_e$ is a summand over abelian groups associated with the source of morphisms ending at e. It is noted that identities in groupoids can be considered as labels morphisms. Hence $(\mathscr{H}\mathcal{A})_e$ is a summand over abelian groups indexed by elements in $\mathrm{costar}_G\,(e)$. Every $E(G)$–map $\psi : \mathcal{A} \to \mathcal{A}'$ gives a family

$\{(\mathscr{H}\psi)_e\}_{e\in E(G)}$ of maps $(\mathscr{H}\psi)_e : (\mathscr{H}\mathcal{A})_e \to (\mathscr{H}\mathcal{A}')_e$. Each map $(\mathscr{H}\psi)_e$ is precisely a collection of maps $\{\psi_{kk^{-1}}\}_{k\in \operatorname{costar} G(e)}$ on summands of $(\mathscr{H}\mathcal{A})_e$. We show that $\mathscr{H}\mathcal{A}$ is a G–module in the following proposition.

Proposition 3.2.1. *Let G be an ordered groupoid and $\mathcal{A}$ an $E(G)$–module. Then $\mathscr{H}\mathcal{A}$ is a G–module.*

Proof. It suffices to show that $\mathscr{H}\mathcal{A}$ admits an $\mathcal{L}(G)$ action. Recall that the ordered groupoid G is a subcategory of $\mathcal{L}(G)$ with a canonical injection $h \mapsto (hh^{-1}, h)$ and every morphism in $\mathcal{L}(G)$ admits a unique decomposition

$$(e, h) = (e, hh^{-1})(hh^{-1}, h)$$

where the first factor is an order morphism in the category $E(G)$ and the second factor is a morphism from the subcategory G. We will thus discuss the action of the morphism (e, h) by considering the actions by the factors.
Let $h \in G(f, z)$ and $\mathcal{A}_{gg^{-1}}$ be a summand in $(\mathscr{H}\mathcal{A})_f$. Then we have the groupoid morphisms

$$\begin{tikzcd} gg^{-1} \arrow[dr, "g"] & & \\ & f \arrow[r, "h"] & z \\ yy^{-1} \arrow[ur, "y"] & & \end{tikzcd}$$

which induces group isomorphisms

$$\begin{tikzcd} \mathcal{A}_{gg^{-1}} \arrow[dr, "\lhd g"] & & \\ & \mathcal{A}_f \arrow[r, "\lhd h"] & \mathcal{A}_z \\ \mathcal{A}_{yy^{-1}} \arrow[ur, "\lhd y"] & & \end{tikzcd}$$

We define the action of h on $a \in \mathcal{A}_{gg^{-1}}$ by

$$a \lhd h = a \in \mathcal{A}_{(gh)(gh)^{-1}} = \mathcal{A}_{gg^{-1}} \ .$$

So h maps $a \in \mathcal{A}_{gg^{-1}}$ to a in the summand of $(\mathscr{H}\mathcal{A})_z$ corresponding to the morphism gh. The summand of $(\mathscr{H}\mathcal{A})_z$ indexed by gh is exactly a copy of $\mathcal{A}_{gg^{-1}}$. The action thus induces a map $(\mathscr{H}\mathcal{A})_f \to (\mathscr{H}\mathcal{A})_z$.

Now suppose $e, f \in E(G)$ and $e \geqslant f$. Then there is an order map $e \to f$. Let $x \in \operatorname{costar}_G(e)$ so that $\mathcal{A}_{xx^{-1}}$ is a summand of $(\mathscr{H}\mathcal{A})_e$. Then there exist a corestriction $(f|x)$ of x to f and by definition of restriction $p = (f|x)\mathbf{d} \leqslant xx^{-1}$. Hence we can find the order map $(xx^{-1}, p) : xx^{-1} \to p$ in $E(G)$. Diagrammatically the order maps are shown as

$$\begin{tikzcd}
xx^{-1} \arrow[r, "x"] \arrow[d, dashed] & e \arrow[d, dashed] \\
p \arrow[r, "(f|x)"] & f
\end{tikzcd}$$

The maps above induces group homomorphisms

$$\begin{tikzcd}
\mathcal{A}_{xx^{-1}} \arrow[r, "\lhd x"] \arrow[d, "\alpha_p^{xx^{-1}}"'] & \mathcal{A}_e \arrow[d, "\alpha_f^e"] \\
\mathcal{A}_p \arrow[r, "\lhd (f|x)"] & \mathcal{A}_f
\end{tikzcd}$$

Thus $\mathcal{A}_p$ is a summand of $(\mathscr{H}\mathcal{A})_f$ indexed by $(f|x)$ and so we define the action of the order map $(e, f) : e \to f$ in $E(G)$ on $a \in \mathcal{A}_{xx^{-1}}$ by

$$a \lhd (e, f) = a\alpha_p^{xx^{-1}} \in \mathcal{A}_p$$

where $\mathcal{A}_p$ is a summand of $(\mathscr{H}\mathcal{A})_f$ indexed by $(f|x)$. It follows that the action gives a morphism $(\mathscr{H}\mathcal{A})_e \to (\mathscr{H}\mathcal{A})_f$.

Therefore the $\mathcal{L}(G)$–action on $\mathscr{H}\mathcal{A}$ is defined as follows. Let $(e, h) \in \mathcal{L}(G)$ and $a \in \mathcal{A}_{xx^{-1}}$ a summand of $(\mathscr{H}\mathcal{A})_e$, then

$$a \lhd (e, h) = (a\alpha_p^{xx^{-1}}) \lhd h = a\alpha_p^{xx^{-1}} \in \mathcal{A}_{((f|x)h)((f|x)h)^{-1}} = \mathcal{A}_p$$

a summand in $(\mathscr{H}\mathcal{A})_z$. This gives a morphism $(\mathscr{H}\mathcal{A})_e \to (\mathscr{H}\mathcal{A})_z$. □

It is clear that $\mathscr{H}\mathcal{A}$ where $\mathcal{A} \in \mathrm{Mod}_{E(G)}$ is a functor $\mathcal{L}(G) \to \mathbf{Ab}$. By the definition of the functor $\mathscr{H}$, we obtain the following lemma.

Lemma 3.2.2. *Let $E(G)$ be the set of identities of an ordered groupoid G. Suppose $\Delta\mathbb{Z} \in \mathrm{Mod}_{E(G)}$. Then $\mathscr{H}\Delta\mathbb{Z}$ is an adjoint module over G.*

We now apply the following corollary to discuss G–maps from $\mathscr{H}\mathcal{A}$ into the adjoint module $\mathbb{Z}G$ over G.

Corollary 3.2.3. *Let $\mathbb{Z}G$ be an adjoint module over the ordered groupoid G and let $\mathcal{A} \in \mathrm{Mod}_{E(G)}$. Then there is an G-map from $\mathscr{H}\mathcal{A}$ to $\mathbb{Z}G$.*

Proof. The main task is to show that there is a natural transformation $\psi : \mathscr{H}\mathcal{A} \to \mathbb{Z}G$ which commutes with the $\mathcal{L}(G)$–actions. The map ψ is necessarily a family of maps $(\psi_f)_{f\in E(G)} : (\mathscr{H}\mathcal{A})_f \to (\mathbb{Z}G)_f$. So the proof is as follows.
Let $\phi : \mathcal{A} \to (\mathbb{Z}G)|_{E(G)}$ be an $E(G)$–map and let the map ψ be the family of maps $(\psi_f)_{f\in E(G)} : (\mathscr{H}\mathcal{A})_f \to (\mathbb{Z}G)_f$. We note that every ψ_f is a collection of maps on summands. So we defined ψ_f by

$$\mathcal{A}_{hh^{-1}}\psi_f = (\mathcal{A}_{hh^{-1}})\phi_{hh^{-1}} = (\mathbb{Z}G)_{hh^{-1}}$$

where $\mathcal{A}_{hh^{-1}}$ is a summand in $(\mathscr{H}\mathcal{A})_f$ and $(\mathbb{Z}G)_{hh^{-1}}$ is the label of $h \in R(G)_f$. We will show that ψ is a natural transformation of the functors $\mathscr{H}\mathcal{A}$ and $\mathbb{Z}G$. Recall that each morphism in $\mathcal{L}(G)$ admits a unique decomposition into a morphism of G and morphism of $E(G)$ considered as categories. So it suffices to split the proof into two cases of showing that ψ commutes with the actions of $\mathcal{L}(G)$.

Case 1. Commutativity of ψ with actions of G.

Suppose z and f are objects of G and let $g \in G(f, z)$. Then the $E(G)$–map ϕ gives the diagram

$$\begin{tikzcd}
\mathcal{A}_{hh^{-1}} \arrow[dr, "\triangleleft\, h"] \arrow[dd, "\phi_{hh^{-1}}"] & & \\
& \mathcal{A}_f \arrow[r, "\triangleleft\, g"] \arrow[dd, "\phi_f"] & \mathcal{A}_z \arrow[dd, "\phi_z"] \\
(\mathbb{Z}G)_{hh^{-1}} \arrow[dr, "\triangleleft\, h"] & & \\
& (\mathbb{Z}G)_f \arrow[r, "\triangleleft\, g"] & (\mathbb{Z}G)_z
\end{tikzcd}$$

By definition

$$\mathcal{A}_{hh^{-1}}\psi_f = (\mathcal{A}_{hh^{-1}})\phi^f_{hh^{-1}} = (\mathbb{Z}G)_{hh^{-1}}$$

where $\mathcal{A}_{hh^{-1}}$ is a summand of $(\mathscr{H}\mathcal{A})_f$ and so the action of the basis element h of $(\mathbb{Z}G)_f$ on $(\mathbb{Z}G)_{hh^{-1}}$ induces the map $(\mathbb{Z}G)_{hh^{-1}} \to (\mathbb{Z}G)_f$. Therefore following with a g action gives

$$(\mathcal{A}_{hh^{-1}}\psi_f) \triangleleft (hg) = ((\mathcal{A}_{hh^{-1}})\phi^f_{hh^{-1}}) \triangleleft (hg) = (\mathbb{Z}G)_{(hg)(hg)^{-1}} \subset (\mathbb{Z}G)_z \ .$$

We note that $(\mathbb{Z}G)_{(hg)(hg)^{-1}}$ is necessarily a copy of $(\mathbb{Z}G)_{hh^{-1}}$ in $(\mathbb{Z}G)_z$ indexed by hg and so we get a morphism $(\mathscr{H}\mathcal{A})_f \to (\mathbb{Z}G)_z$. Also, recall that the action of G on a summand gives the a copy of the summand but with a shift in index. Thus for $\mathcal{A}_{hh^{-1}}$ a summand of $(\mathscr{H}\mathcal{A})_f$, the action $\mathcal{A}_{hh^{-1}} \triangleleft g = \mathcal{A}_{(hg)(hg)^{-1}}$ which is a copy of $\mathcal{A}_{hh^{-1}}$ in $(\mathscr{H}\mathcal{A})_z$ indexed by hg. Thus

$$(\mathcal{A}_{hh^{-1}} \triangleleft g)\psi_z = \mathcal{A}_{hh^{-1}}\phi^z_{hh^{-1}} = (\mathbb{Z}G)_{hh^{-1}}$$

considered as a summand of $(\mathbb{Z}G)_z$. The action of the basis element hg of $(\mathbb{Z}G)_z$ yields

$$(\mathcal{A}_{hh^{-1}} \triangleleft g)\psi_z = (\mathcal{A}_{hh^{-1}}\phi^z_{hh^{-1}}) \triangleleft (hg) = (\mathbb{Z}G)_{(hg)(hg)^{-1}} \subset (\mathbb{Z}G)_z$$

and so induces a map $(\mathbb{Z}G)_{hh^{-1}} \to (\mathbb{Z}G)_z$. Thus we obtain a morphism

$(\mathscr{H}\mathcal{A})_f \to (\mathbb{Z}G)_z$. Hence we get that the diagram

$$\begin{tikzcd}
(\mathscr{H}\mathcal{A})_f \supset \mathcal{A}_{hh^{-1}} \arrow[r, "\phi^f_{hh^{-1}}"] \arrow[d, "\lhd g"'] & (\mathbb{Z}G)_{hh^{-1}} \arrow[r, "\lhd h"] & (\mathbb{Z}G)_f \arrow[d, "\lhd g"] \\
(\mathscr{H}\mathcal{A})_z \supset \mathcal{A}_{hh^{-1}} \arrow[r, "\phi^z_{hh^{-1}}"'] & (\mathbb{Z}G)_{hh^{-1}} \arrow[r, "\lhd hg"'] & (\mathbb{Z}G)_z
\end{tikzcd}$$

commutes. Therefore for every $g \in G(f,z)$, the diagram

$$\begin{tikzcd}
(\mathscr{H}\mathcal{A})_f \arrow[r, "\psi_f"] \arrow[d, "\lhd g"] & (\mathbb{Z}G)_f \arrow[d, "\lhd g"] \\
(\mathscr{H}\mathcal{A})_z \arrow[r, "\psi_z"'] & (\mathbb{Z}G)_z
\end{tikzcd}$$

commutes.

Case 2. Commutativity of ψ with actions of $E(G)$.

Now let e and f be objects of G such that $e \geqslant f$. The action of the order map $(e,f) \in E(G)$ on $(\mathbb{Z}G)_e$ gives a homomorphism $\beta^e_f : (\mathbb{Z}G)_e \to (\mathbb{Z}G)_f$. Let $\mathcal{A}_{xx^{-1}}$ be a summand of $(\mathscr{H}\mathcal{A})_e$. Then by definition $\mathcal{A}_{xx^{-1}}\psi_e = (\mathcal{A}_{xx^{-1}})\phi^e_{xx^{-1}} = (\mathbb{Z}G)_{xx^{-1}}$ and since the action of the basis element x in $(\mathbb{Z}G)_e$ gives the homomorphism $(\mathbb{Z}G)_{xx^{-1}} \to (\mathbb{Z}G)_e$ following with the action of the order map (e,f), we get

$$(\mathcal{A}_{xx^{-1}}\psi_e) \lhd (e,f) = ((\mathcal{A}_{xx^{-1}})\phi^e_{xx^{-1}}) \lhd (x\beta^e_f) = (\mathbb{Z}G)_{(f|x)(f|x)^{-1}} \subset (\mathbb{Z}G)_f \ .$$

This gives a morphism $(\mathscr{H}\mathcal{A})_e \to (\mathbb{Z}G)_f$. Also the action of the order map $(e,f) \in E(G)$ on $(\mathscr{H}\mathcal{A})_e$ gives a homomorphism $\alpha^e_f : (\mathscr{H}\mathcal{A})_e \to (\mathscr{H}\mathcal{A})_f$. Suppose $\mathcal{A}_{xx^{-1}}$ is a summand of $(\mathscr{H}\mathcal{A})_e$ where $p = (f|x)\mathbf{d}$. Then $\mathcal{A}_{xx^{-1}} \lhd (e,f) = \mathcal{A}_{xx^{-1}} \cdot \alpha^{xx^{-1}}_p = \mathcal{A}_p$ a summand of $(\mathscr{H}\mathcal{A})_f$. Therefore applying ψ followed by the action by $(f|x)$ gives

$$(\mathcal{A}_{xx^{-1}} \lhd (e,f))\psi_p = (\mathcal{A}_{xx^{-1}} \cdot \alpha^{xx^{-1}}_p \phi^f_p) \lhd (f|x) = (\mathbb{Z}G)_{(f|x)(f|x)^{-1}} \subset (\mathbb{Z}G)_f$$

since $p = (f|x)(f|x)^{-1}$. This gives a morphism $(\mathscr{H}\mathcal{A})_e \to (\mathbb{Z}G)_f$. From the arguments above we get the diagram

$$\begin{tikzcd}
(\mathscr{H}\mathcal{A})_e \supset A_{xx^{-1}} \arrow[r, "\phi_{xx^{-1}}"] \arrow[d, "\alpha_p^{xx^{-1}}"] & (\mathbb{Z}G)_{xx^{-1}} \arrow[r, "\lhd x"] \arrow[d, "\beta_p^{xx^{-1}}"] & (\mathbb{Z}G)_{xx^{-1}} \subset (\mathbb{Z}G)_e \arrow[d, "\beta_f^e"] \\
(\mathscr{H}\mathcal{A})_f \supset A_p \arrow[r, "\phi_p"'] & (\mathbb{Z}G)_p \arrow[r, "\lhd(f|x)"'] & (\mathbb{Z}G)_{(f|x)(f|x)^{-1}} \subset (\mathbb{Z}G)_f
\end{tikzcd}$$

The left diagram commutes since ϕ is an $E(G)$-map. In $\mathcal{L}(G)$ the morphism $(xx^{-1}, x)(e, f) = (xx^{-1}, p)(p, (f|x))$ hence their actions are the same thus the inner right diagram commute. The commutative of the outer diagram is a consequence of the commutativity of the two inner diagrams. This shows that diagram

$$\begin{tikzcd}
(\mathscr{H}\mathcal{A})_e \arrow[r, "\psi_e"] \arrow[d, "\alpha_f^e"] & (\mathbb{Z}G)_e \arrow[d, "\beta_f^e"] \\
(\mathscr{H}\mathcal{A})_f \arrow[r, "\psi_f"'] & (\mathbb{Z}G)_f
\end{tikzcd}$$

commutes.

Therefore ψ commutes with the actions of morphisms of $\mathcal{L}(G)$ and so it is a G-map as desired. $\square$

3.2.3 Adjunction of functors

In the sequel we show that the functor $\mathscr{H}$ discussed so far is left adjoint to the restriction $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{E(G)}$. Our account in the following paragraph is adapted from [29]. We use relevant ideas from the theory of ordered groupoids to provide detailed verification of the concepts adapted from [29] by Loganathan for the associated inverse semigroups.

Proposition 3.2.4. *Let G be an ordered groupoid and let $\mathscr{H}$ be the functor $\mathrm{Mod}_{E(G)} \to \mathrm{Mod}_{\mathcal{L}(G)}$ defined by $(\mathscr{H}\mathcal{A})_e = \bigoplus_{g \in R(G)_e} \mathcal{A}_{gg^{-1}}$ where $\mathcal{A}$ is an $E(G)$–module. Then $\mathscr{H}$ is left adjoint to the restriction $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{E(G)}$.*

Proof. Let $\mathcal{B}$ be a module over the ordered groupoid G and let $\mathcal{A} \in \mathrm{Mod}_{E(G)}$. Then the main task is to show that there is a natural bijection

$$\mathrm{Mod}_{\mathcal{L}(G)}(\mathscr{H}\mathcal{A}, \mathcal{B}) \cong \mathrm{Mod}_{E(G)}(\mathcal{A}, \mathcal{B}|_{E(G)}) \ .$$

Let $\phi : \mathcal{A} \to \mathcal{B}|_{E(G)}$ be an $E(G)$–map. Then a G–map $\psi : \mathscr{H}\mathcal{A} \to \mathcal{B}$ is necessarily a collection of maps $\psi_e : (\mathscr{H}\mathcal{A})_e \to \mathcal{B}_e$ for all $e \in E(G)$. Define ψ_e by

$$(\mathscr{H}\mathcal{A})_e\psi_e = \mathcal{A}_e\phi_e = \mathcal{B}_e \ .$$

Then it is clear that ψ is a G–map and so the $E(G)$–map ϕ determines the G–map ψ. Thus distinct G–maps ψ are defined via distinct ϕ. So we get an injection

$$\Upsilon : \mathrm{Mod}_{\mathcal{L}(G)}(\mathscr{H}\mathcal{A}, \mathcal{B}) \to \mathrm{Mod}_{E(G)}(\mathcal{A}, \mathcal{B}|_{E(G)}). \tag{3.2.1}$$

The injection Υ sends the morphism ψ_e to ϕ_e. Now define the map

$$\Upsilon^* : \mathrm{Mod}_{E(G)}(\mathcal{A}, \mathcal{B}|_{E(G)}) \to \mathrm{Mod}_{\mathcal{L}(G)}(\mathscr{H}\mathcal{A}, \mathcal{B})$$

by $(\phi)\Upsilon^* = \psi$. So that Υ^* precisely sends ϕ_e to ψ_e. Then $\Upsilon^*\Upsilon$ is the identity map on ϕ. Also the composition $\Upsilon\Upsilon^*$ sends ψ to ψ and so Υ^* and Υ are inverses to each other. Hence Υ is a canonical bijection between $\mathrm{Mod}_{\mathcal{L}(G)}(\mathscr{H}\mathcal{A}, \mathcal{B})$ and $\mathrm{Mod}_{E(G)}(\mathcal{A}, \mathcal{B}|_{E(G)})$. Hence $\mathscr{H}$ is left adjoint to the restriction $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{E(G)}$ as desired. □

3.3 Homology of the semilattice of idempotents

In this section we discuss the concept of (co)homology of ordered groupoids and that of its set of identities. The homology of arbitrary small categories has been treated in the following recommended literature: [15], [7] and [8]. The details in the sequel are presented to match our interest, analogous to discussions on the homology of inverse semigroups in [29], and [25]. Our discussions so far has treated the preliminary concepts to describing the idea of (co)homology of ordered groupoids and that of their set of identities. We shall discuss the (co)homology of ordered groupoids and infer that of their set of identities with appropriate modifications. We proceed with the definition of the homology functor from the module category into the category of abelian groups. A similar concept to the homology functor is the derived functor of a functor on abelian categories (see [7], [15] and [8]). Our arguments here are adapted from [7].

Definition Let $\mathscr{C}$ be an abelian category and $\mathfrak{J}$ an object of $\mathscr{C}$. Then $\mathfrak{J}$ is *injective* if for any monomorphism $\mathcal{A}' \xrightarrow{\mu} \mathcal{A}$ and any morphism $\mathcal{A}' \xrightarrow{\tau} \mathfrak{J}$ there exist a morphism $\mathcal{A} \xrightarrow{\alpha} \mathfrak{J}$ such that $\mu\alpha = \tau$.

Definition Let $\mathscr{C}$ be an abelian category and P an object of $\mathscr{C}$. Then we say that P is *projective* if for any epimorphism $\mathcal{A} \xrightarrow{\mu} \mathcal{A}'$ and any morphism $P \xrightarrow{\tau} \mathcal{A}'$ there exist a morphism $P \xrightarrow{\alpha} \mathcal{A}$ such that $\alpha\mu = \tau$.

Lemma 3.3.1. $\mathrm{Mod}_{\mathcal{L}(G)}$ *is an abelian category with enough projectives and injectives.*

An exact sequence $\cdots \to P_i \to \cdots \to P_0$ of projective objects together with an isomorphism $P_0 \to \mathcal{A}$ is called a *projective resolution* of $\mathcal{A}$. The fact that $\mathrm{Mod}_{\mathcal{L}(G)}$ has enough projectives implies that any object of $\mathrm{Mod}_{\mathcal{L}(G)}$ admits a projective resolution of length at least one. Let $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}_{\mathcal{L}(G)}$ and denote by $\mathcal{A} \otimes B$ the module defined by $(\mathcal{A} \otimes \mathcal{B})_e = \mathcal{A}_e \otimes \mathcal{B}_e$. Suppose P is a projective resolution of $\mathcal{A}$. Then we define

$$\mathrm{Ext}^n_{\mathcal{L}(G)}(\mathcal{A}, \mathcal{B}) = H^n(\mathrm{Hom}_{\mathcal{L}(G)}(P, \mathcal{B}))$$
$$\mathrm{Tor}^{\mathcal{L}(G)}_n(\mathcal{A}, \mathcal{B}) = H_n(\varinjlim{}^{\mathcal{L}(G)} P \otimes \mathcal{B}) \ .$$

Suppose $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$. We define the nth *homology group* of G with coefficients in $\mathcal{A}$ by

$$H_n(G, \mathcal{A}) = \mathrm{Tor}^G_n(\Delta\mathbb{Z}, \mathcal{A}) \ .$$

The following proposition on the homology functor is adapted from [7].

Proposition 3.3.2. *Suppose G is an ordered groupoid and let $\mathcal{A}$ be a G–module. Then the homology functor $H : \mathrm{Mod}_{\mathcal{L}(G)} \to \boldsymbol{Ab}$ satisfies*

1. $H_0(G, \mathcal{A}) = \varinjlim^{\mathcal{L}(G)} \mathcal{A}$.

2. $H_n(G, \mathcal{A}) = 0$ *for all* $n \geqslant 1$ *for all projective* $\mathcal{A}$.

Am immediate observation is the following.

Corollary 3.3.3. *Let G be an ordered groupoid with poset of identities $E(G)$. Suppose $\mathcal{A} \in \mathrm{Mod}_{E(G)}$. Then the homology functor $H : \mathrm{Mod}_{E(G)} \to \boldsymbol{Ab}$ satisfies*

1. $H_0(E(G), \mathcal{A}) = \varinjlim^{E(G)} \mathcal{A}$,

2. $H_n(E(G), \mathcal{A}) = 0$ *for all* $n \geqslant 1$ *for all projective* $\mathcal{A}$.

Theorem 3.3.4. *Let G be an ordered groupoid and $\mathcal{A}$ a G–module. Then there are natural isomorphisms $H_n(E(G), \mathcal{A}) \cong \mathrm{Tor}_n^{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$.*

Proof. Proposition 3.2.4 shows that the functor $\mathscr{H} : \mathrm{Mod}_{E(G)} \to \mathrm{Mod}_{\mathcal{L}(G)}$ is left adjoint to the restriction $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{E(G)}$. Hence $\mathscr{H}$ is right exact and it follows from standard argument that $\mathscr{H}$ preserves epimorphisms (projectives). The category $\mathrm{Mod}_{E(G)}$ has enough projectives, a consequence of Lemma 3.3.1 and so every $E(G)$–module admits a projective resolution of length at least one. Thus the constant module $\Delta\mathbb{Z}$ possesses a projective resolution. Let P be a projective resolution of $\Delta\mathbb{Z}$. But the nth homology group of $E(G)$ with coefficient in $\mathcal{A}$ is defined by $H_n(E(G), \mathcal{A}) = \mathrm{Tor}_n^{E(G)}(\Delta\mathbb{Z}, \mathcal{A})$. But

$$\mathrm{Tor}_n^{E(G)}(\Delta\mathbb{Z}, \mathcal{A}) = H_n(\varinjlim^{E(G)} P, \mathcal{A})$$

and so $H_n(E(G), \mathcal{A}) = H_n(\varinjlim^{E(G)} P, \mathcal{A})$. Now since the functor $\mathscr{H}$ preserves epimorphisms, $\mathscr{H}P$ is projective resolution of $\mathscr{H}\Delta\mathbb{Z}$. By lemma 3.2.2, we have $\mathscr{H}\Delta\mathbb{Z} = \mathbb{Z}G$ and so $\mathscr{H}P$ is a projective resolution of $\mathbb{Z}G$. Thus we have

$$H_n(\varinjlim^{\mathcal{L}(G)} \mathcal{H}P, \mathcal{A}) = \mathrm{Tor}_n^{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$$

and hence the isomorphisms

$$H_n(E(G), \mathcal{A}) = H_n(\varinjlim^{E(G)} P, \mathcal{A}) \cong H_n(\varinjlim^{\mathcal{L}(G)} \mathcal{H}P, \mathcal{A}) = \mathrm{Tor}_n^{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$$

as desired. □

We spend the sequel in discussing the *cohomology* functor, the dual notion of homology functor on ordered groupoids and that of its set of identities. The ideas presented here are key to explaining the main result of this chapter.

Let $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$. We define the nth *cohomology group* of G with coefficients in $\mathcal{A}$ by

$$H_n(G, \mathcal{A}) = \mathrm{Ext}^n_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \ .$$

Proposition 3.3.5. *Let G be an ordered groupoid and $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$. Then the universal cohomology functor on G with values in* ***Ab*** *satisfies*

1. $H^0(G, \mathcal{A}) = \varprojlim^{\mathcal{L}(G)} \mathcal{A}$.
2. $H^n(G, \mathcal{A}) = 0$ *for* $n > 0$ *and* $\mathcal{A}$ *injective.*

The above proposition follows from [7].

Corollary 3.3.6. *Let G be an ordered groupoid with set of identities $E(G)$. Suppose $\mathcal{A} \in \mathrm{Mod}_{E(G)}$. Then the universal cohomology functor on $E(G)$ with values in* ***Ab*** *satisfies the following conditions,*

1. $H^0(E(G), \mathcal{A}) = \varprojlim^{E(G)} \mathcal{A}$.
2. $H^n(E(G), \mathcal{A}) = 0$ *for* $n > 0$ *and for all* $\mathcal{A}$ *injective.*

We make the following identification of the cohomology of the set of identities of an ordered groupoid which will later be used in presenting the connection between the cohomology of an ordered groupoid with that of its set of identities. The idea is inspired by [29].

Theorem 3.3.7. *Suppose G is an ordered groupoid and that $\mathcal{A}$ is a G-module. Then there are natural isomorphisms $H^n(E(G), \mathcal{A}) \cong \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$.*

Proof. The module categories have enough projectives and so every module admits a projective resolution. Let $\Delta\mathbb{Z}$ be a constant $E(G)$-module and choose a projective resolution P of $\Delta\mathbb{Z}$. Then the cohomology of $E(G)$ with coefficient in $\mathcal{A}$ is defined by

$$H^n(E(G), \mathcal{A}) = \mathrm{Ext}^n_{E(G)}(\Delta\mathbb{Z}, \mathcal{A}) \ .$$

But

$$\mathrm{Ext}^n_{E(G)}(\Delta\mathbb{Z}, \mathcal{A}) = H^n\left(\mathrm{Hom}_{E(G)}(P, \mathcal{A})\right)$$

hence

$$H^n(E(G), \mathcal{A}) = H^n\left(\mathrm{Hom}_{E(G)}(P, \mathcal{A})\right) \ .$$

Now $\mathscr{H}$ is left adjoint to the restriction from Proposition 3.2.4 and so preserves projectives. This implies $\mathscr{H}P$ is a projective resolution of $\mathscr{H}\Delta\mathbb{Z}$. We define

$$\mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) = \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathscr{H}\Delta\mathbb{Z}, \mathcal{A})$$

since $\mathbb{Z}G = \mathscr{H}\Delta\mathbb{Z}$ from lemma 3.2.2. We obtain

$$\mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) = \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathscr{H}\Delta\mathbb{Z}, \mathcal{A}) = H^n\left(\mathrm{Hom}_{\mathcal{L}(G)}(\mathcal{H}P, \mathcal{A})\right) \ .$$

Therefore the isomorphism

$$H^n(E(G), \mathcal{A}) = H^n\left(\mathrm{Hom}_{E(G)}(P, \mathcal{A})\right) \cong H^n\left(\mathrm{Hom}_{\mathcal{L}(G)}(\mathscr{H}P, \mathcal{A})\right) = \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$$

as desired $\square$

3.4 Cohomology of ordered groupoids with an adjoined identity

In this section we begin with a discussion on the idea of constructing ordered groupoids G^I with an adjoined identity. The ordered groupoid G^I made a brief appearance in [33] where the connection between the second cohomology and the set of congruence classes of extensions with abelian kernel of ordered groupoids was studied using the concept of factor sets of ordered groupoids. A similar study was carried out by Lausch in [25] to put in greater generality the findings of D'Alarcao [13] and Coudron [9] on inverse semigroups. Loganathan in [30] uses the approach of adjoining an identity to regular semigroups to account for the connection between the cohomology and extensions of regular semigroups. We present some functors on the module categories of ordered groupoids with an adjoined identity and that of the ordered groupoid and show that the functors are adjoint pairs. We explain some connections between the cohomology of an ordered groupoid with an adjoined identity and that of the ordered groupoid. We commence with the discussion of the construction of an ordered groupoid with an adjoined identity.

Ordered groupoids with an adjoined identity

Let G be an ordered groupoid and set $G^I = G \cup \{I\}$ where I is a symbol and $I \notin G$. We make G^I an ordered groupoid by extending the data of the ordered groupoid G onto G^I as follows. Set $e \leqslant I$ for every object e of G. The restriction of I to an object e is identity morphism at e. It is easy to see that G^I is an ordered groupoid under these definitions together with the data of the ordered groupoid G. The maximal ordered subgroupoids of G^I are G and the singleton groupoid $\{I\}$. The associated category of G^I is defined by the following data. The object set of the category $\mathcal{L}(G^I)$ is $E(G) \cup \{I\}$. Morphisms of $\mathcal{L}(G^I)$ comprise morphisms of G and the order maps $\alpha_e^I : I \to e$ for all $e \in E(G)$ inherited from the extension of the ordering on G. Every morphism $g \in G(e, f)$ is a morphism in a commutative triangle

$$\begin{tikzcd} & I \arrow[dl, "\alpha_e^I"'] \arrow[dr, "\alpha_f^I"] & \\ e \arrow[rr, "g"] & & f \end{tikzcd}$$

3.4.1 Adjunction of functors on module categories

The inclusion $G \to G^I$ induces the restriction $\mathrm{Mod}_{\mathcal{L}(G^I)} \to \mathrm{Mod}_{\mathcal{L}(G)}$. The restriction is a covariant functor that associates each $\mathcal{A}^I \in \mathrm{Mod}_{\mathcal{L}(G^I)}$ the G-module $\mathcal{A}$. That is the restriction associates each G^I module with the module over the maximal ordered subgroupoid G of G^I. We will later show that the restriction has a right adjoint.

Let $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$. We extend $\mathcal{A}$ into $\mathcal{A}^I$ as follows. Define $\mathcal{A}^I$ by

$$\mathcal{A}^I_e = \begin{cases} \mathcal{A}_e & \text{if } e \neq I \\ \varprojlim^{E(G)} \mathcal{A} & \text{if } e = I \end{cases}$$

where $e \in E(G^I)$.

Proposition 3.4.1. *Let G be an ordered groupoid and $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$. Then $\mathcal{A}^I$ is a G^I–module.*

Proof. It is clear that $\mathcal{A}^I$ is a functor $\mathcal{L}(G^I) \to \mathbf{Ab}$ and so it suffices to show that $\mathcal{L}(G^I)$ acts on A^I. To explain the $\mathcal{L}(G^I)$ action, recall that morphisms in $\mathcal{L}(G^I)$ decomposes uniquely into a composite of a G^I morphism and an $E(G^I)$ morphism. So it is enough to discuss the action by G^I and and $E(G^I)$.

We explain the G^I action on $\mathcal{A}^I$ as follows. It is noted that the G^I morphisms are exactly the G morphisms. Let f and z be objects of G and $g \in G(f, z)$. Then the action of g on $a \in \mathcal{A}_f$ is given by $a \triangleleft g \in \mathcal{A}_z$. This gives a map $\mathcal{A}_f \to \mathcal{A}_z$. The map is necessarily a group isomorphism since g has an inverse action obtained from g^{-1}. Now we have that in $E(G^I)$, there are order morphisms corresponding to order morphisms in $E(G)$ and order morphisms inherited from the extension of the natural order in G to G^I. Let e and f be objects in G such that $e \geqslant f$. Then there is a unique order map $(e, f) : e \to f$ and the action of the order map is written $a \triangleleft (e, f) \in \mathcal{A}_f$ for $a \in \mathcal{A}_e$. This gives a homomorphism $\alpha^e_f : \mathcal{A}_e \to \mathcal{A}_f$. Suppose e is an object of G. Then the extension of the order map gives an order map $(I, e) : I \to e$. The action of (I, e) on $a \in \mathcal{A}^I_I$ is given by $a \triangleleft (I, e) = a\pi_e \in \mathcal{A}_e$ where π_e is the canonical projection $\varprojlim^{E(G)} \mathcal{A} \to \mathcal{A}_e$. Thus the induced map $\alpha^I_e : \mathcal{A}^I_I \to \mathcal{A}_e$ is exactly the projection.

Therefore the action of $\mathcal{L}(G^I)$ on $\mathcal{A}^I$ is given by

$$a \lhd (e,g) = \begin{cases} (a \lhd (e, gg^{-1})) \lhd g \in \mathcal{A}_{g^{-1}g} & \text{if } e \neq I \\ (a\pi_{gg^{-1}}) \lhd g \in \mathcal{A}_{g^{-1}g} & \text{if } e = I \end{cases}$$

where $a \in \mathcal{A}_e$. □

Let $U : \mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{\mathcal{L}(G^I)}$ be a functor of module categories defined by $\mathcal{A} \mapsto \mathcal{A}^I$. We show that U is right adjoint to the restriction.

Lemma 3.4.2. *Suppose G^I be an ordered groupoid obtained from the ordered groupoid G and let $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$. The functor $U : \mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{\mathcal{L}(G^I)}$ defined by $\mathcal{A} \mapsto \mathcal{A}^I$ is right adjoint to the restriction* $\mathrm{Mod}_{\mathcal{L}(G^I)} \to \mathrm{Mod}_{\mathcal{L}(G)}$.

Proof. Suppose $\mathcal{B}$ is a G^I–module and let $\mathcal{A}$ be a G–module. Then it suffices to show that there is a canonical bijection $\mathrm{Mod}_{\mathcal{L}(G^I)}(\mathcal{B}, U\mathcal{A}) \to \mathrm{Mod}_{L(G)}(\mathcal{B}|_{\mathcal{L}(G)}, \mathcal{A})$. Let $\phi : \mathcal{B}|_{\mathcal{L}(G)} \to \mathcal{A}$ be a G-map. Then ϕ is necessarily a family of maps $\phi_e : \mathcal{B}_e \to \mathcal{A}_e$ and commutes with the actions of morphisms of $\mathcal{L}(G)$. So if $g \in G(f,z)$ and $(e,f) : e \to f$ is the unique order map corresponding to $e \geqslant f$ in $E(G)$. Then we have that the diagram

$$\begin{tikzcd} \mathcal{B}_e \arrow[r, "\phi_e"] \arrow[d, "\alpha^e_f"] \arrow[dd, bend right=60, "\lhd(e,g)"'] & \mathcal{A}_e \arrow[d, "\alpha^e_f"] \\ \mathcal{B}_f \arrow[r, "\phi_f"] \arrow[d, "\lhd g"] & \mathcal{A}_f \arrow[d, "\lhd g"] \\ \mathcal{B}_z \arrow[r, "\phi_z"] & \mathcal{A}_z \end{tikzcd}$$

commute.

Suppose $\varphi : \mathcal{B} \to U\mathcal{A}$ is a G^I–map. Then ϕ is the restriction of φ and so ϕ determines φ when restricted to the subgroupoid G. In precise terms, the maps φ_e are determined by ϕ_e for all $e \in E(G)$. It therefore suffice to examine the map $\varphi_I : \mathcal{B}_I \to (U\mathcal{A})_I$ to completely describe the G^I–map φ. Consider composition of φ with actions of morphisms (I,e) in $\mathcal{L}(G^I)$. Since φ is a natural transformation, the diagram

$$\begin{array}{ccc} \mathcal{B}_I & \xrightarrow{\varphi_I} & (U\mathcal{A})_I = \varprojlim^{E(G)}\mathcal{A} \\ \Big\downarrow{\scriptstyle \lhd(I,e)} & & \Big\downarrow{\scriptstyle \lhd(I,e)} \\ \mathcal{B}_e & \xrightarrow{\varphi_e} & \mathcal{A}_e \end{array}$$

must commute. We note that the map φ_e is determined by ϕ_e. Also, the action of (I, e) determines the map $\mathcal{B}_I \to \varprojlim^{E(G)}\mathcal{B}$ and φ_e determines the map $\varprojlim^{E(G)}\mathcal{A} \to \mathcal{B}_e$ and hence induces $\varprojlim^{E(G)}\mathcal{B} \to \varprojlim^{E(G)}\mathcal{A}$. Therefore the map φ_I is determined by ϕ. So there is an injection

$$\rho : \mathrm{Mod}_{\mathcal{L}(G^I)}(\mathcal{B}, U\mathcal{A}) \to \mathrm{Mod}_{L(G)}(\mathcal{B}|_{\mathcal{L}(G)}, \mathcal{A})$$

defined by $(\varphi) \mapsto (\phi)$. The map

$$\rho' : \mathrm{Mod}_{L(G)}(\mathcal{B}|_{\mathcal{L}(G)}, \mathcal{A}) \to \mathrm{Mod}_{\mathcal{L}(G^I)}(\mathcal{B}, U\mathcal{A})$$

defined by $(\phi) \mapsto (\varphi)$ is the inverse of ρ as $\rho\rho'$ and $\rho'\rho$ are identity maps. Hence ρ is a canonical bijection $\mathrm{Mod}_{\mathcal{L}(G^I)}(\mathcal{B}, U\mathcal{A}) \cong \mathrm{Mod}_{L(G)}(\mathcal{B}|_{\mathcal{L}(G)}, \mathcal{A})$ as desired. $\square$

Given a G–module $\mathcal{A}$, let $\mathcal{A}^0$ be a functor $\mathcal{A}^0 : \mathcal{L}(G^I) \to \mathbf{Ab}$ that associates every $e \in E(G)$ with the abelian group $\mathcal{A}_e$ and I with the singleton group 0. It is evident that $\mathcal{A}^0$ admits an $\mathcal{L}(G)$ action. We make $\mathcal{A}^0$ a G^I–module by extending the $\mathcal{L}(G)$ action to an $\mathcal{L}(G^I)$ action by the following definitions. Let $\alpha_e^0 : 0 \to \mathcal{A}_e$ be the group homomorphism induced by the unique order map from $I \geqslant e$ in $E(G^I)$. Then action of the order map on $a \in \mathcal{A}_I^0$ is given by

$$a \lhd (I, e) = 0 \in \mathcal{A}_e\ .$$

Therefore $\mathcal{A}^0$ is a G^I–module. It is a routine verification to show the functor $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{\mathcal{L}(G^I)}$ which associates $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$ with the module $\mathcal{A}^0$ is left adjoint to the restriction $\mathrm{Mod}_{\mathcal{L}(G^I)} \to \mathrm{Mod}_{\mathcal{L}(G)}$.

3.4.2 Cohomology of ordered groupoids with an adjoined identity

Let $\mathcal{A}^I \in \mathrm{Mod}_{\mathcal{L}(G^I)}$. We define the nth *cohomology group* of G^I with coefficients in $\mathcal{A}^I$ by

$$H^n(G^I, \mathcal{A}^I) = \mathrm{Ext}^n_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}^I)\ .$$

We devote the rest of the subsection to discuss some general results about cohomology groups of ordered groupoids. We spend the following paragraphs to present some exact sequences of cohomology groups and make some identifications of cohomology groups necessary for discussing the main result of this chapter.

Let G be an ordered groupoid. Consider the augmentation map $\mathbb{Z}G \to \Delta\mathbb{Z}$ with augmentation ideal KG. The embedding and the augmentation map yield the exact sequence $KG \to \mathbb{Z}G \to \Delta\mathbb{Z}$ of G-modules. Suppose $\mathcal{A}$ is a G–module. We use the contravariant $\mathrm{Ext}_{\mathcal{L}(G)}(-, \mathcal{A})$ functor to generate the long exact sequence

$$\cdots \to \mathrm{Ext}^n_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \to \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \to \mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A}) \to \mathrm{Ext}^{n+1}_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \to \cdots$$

by applying it to the short exact sequence $0 \to KG \to \mathbb{Z}G \to \Delta\mathbb{Z}$. By definition, $\mathrm{Ext}^0_{\mathcal{L}(G)}(-, \mathcal{A}) = \mathrm{Hom}_{\mathcal{L}(G)}(-, \mathcal{A})$ and so we obtain the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(KG, \mathcal{A}) \to \mathrm{Ext}^1_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \\ &\to \mathrm{Ext}^1_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \to \mathrm{Ext}^1_{\mathcal{L}(G)}(KG, \mathcal{A}) \to \mathrm{Ext}^2_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \to \cdots \qquad (3.4.1) \end{aligned}$$

It is noted that $\mathrm{Ext}^n_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) = H^n(\mathcal{L}(G), \mathcal{A})$.

Now let G^I be an ordered groupoid obtained from G by adjoining the symbol I to G. Suppose $\Delta\mathbb{Z}$ is the constant G^I–module. Then the epimorphism $\mathbb{Z}G^I \to \Delta\mathbb{Z}$ together with the embedding of its kernel KG^I yields the short exact sequence $KG^I \to \mathbb{Z}G^I \to \Delta\mathbb{Z}$ of G^I–modules. Let $\mathcal{A}^0$ be a G^I–module. Applying the contravariant functor $\mathrm{Ext}_{\mathcal{L}(G^I)}(-, \mathcal{A}^0)$ to the short exact sequence yields the long exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \mathrm{Ext}^i_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) \to$$
$$\mathrm{Ext}^i_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to \mathrm{Ext}^{i+1}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \cdots$$

We have that $\mathrm{Ext}^0_{\mathcal{L}(G^I)}(-, \mathcal{A}^0) = \mathrm{Hom}_{\mathcal{L}(G)}(-, \mathcal{A}^0)$ and so we obtain the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \mathrm{Hom}_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) \to \mathrm{Hom}_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to$$
$$\mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to \cdots \quad (3.4.2)$$

We note that $\mathrm{Ext}^n_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) = H^n(\mathcal{L}(G^I), \mathcal{A}^0)$

Remark 3.4.1 The augumentation ideal KG^I consists of the union of

- $\ker\big((\mathbb{Z}G^I)_e \to \mathbb{Z}\ (= (\Delta\mathbb{Z})_e\big) = KG$ for $e \in G_0$, and
- $\ker\big((\mathbb{Z}G^I)_I \to \mathbb{Z}\ (\ = (\Delta\mathbb{Z})_I\big)$.

However we recall that $(\mathbb{Z}G^I)_I$ is generated by the identity map on the symbol I and so $(\mathbb{Z}G^I)_I = \mathbb{Z}$. Therefore $\ker\big((\mathbb{Z}G^I)_I \to \mathbb{Z}\big) = \ker(\mathbb{Z} \to \mathbb{Z}) = 0$. Therefore

$$KG^I = KG \cup 0 = (KG)^0$$

where $(KG)^0$ is the G^I-module obtained by adding 0 at the idempotent I. We infer from the remark that

$$\mathrm{Ext}^n_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) = \mathrm{Ext}^n_{\mathcal{L}(G^I)}((KG)^0, \mathcal{A}^0) = \mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A}) \quad (3.4.3)$$

The isomorphisms in (3.4.3) leads to the identifications

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathcal{L}(G)}(KG, \mathcal{A}) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{i+1}_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{L}(G)}(KG, \mathcal{A}) \arrow[d, equal] \\
\mathrm{Ext}^i_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0)
\end{tikzcd}$$

in (3.4.1) and (3.4.2). Following the identification in dimension zero we concatenate with the bottom sequence and obtain

$$\begin{tikzcd}
& \mathrm{Ext}^0_{\mathcal{L}(G)}(\Delta\mathbb{Z},\mathcal{A}) \arrow[r] & \mathrm{Ext}^0_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A}) \arrow[r] & \mathrm{Ext}^0_{\mathcal{L}(G)}(KG,\mathcal{A}) \arrow[dll, equal] & \\
& \mathrm{Ext}^0_{\mathcal{L}(G^I)}(KG^I,\mathcal{A}^0) \arrow[r] & \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0) \arrow[r] & \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I,\mathcal{A}^0) \arrow[r] & \cdots
\end{tikzcd}$$

This is precisely the sequence

$$\begin{aligned}
&0 \to \mathrm{Hom}_{\mathcal{L}(G)}(\Delta\mathbb{Z},\mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(KG,\mathcal{A}) \xrightarrow{=} \mathrm{Hom}_{\mathcal{L}(G^I)}(KG^I,\mathcal{A}^0)\\
&\to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I,\mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(KG^I,\mathcal{A}^0) \to \mathrm{Ext}^2_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0) \to \cdots
\end{aligned}$$

Let δ be the composite map

$$\mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(KG,\mathcal{A}) \xrightarrow{=} \mathrm{Hom}_{\mathcal{L}(G^I)}(KG^I,\mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0)\ .$$

Using δ we obtain the following.

Proposition 3.4.3. *Let G^I be an ordered groupoid obtained from the ordered groupoid G and let $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$ and $\mathcal{A}^0 \in \mathrm{Mod}_{\mathcal{L}(G^I)}$. Then the* Hom –Ext *sequence*

$$\begin{aligned}
&0 \to \mathrm{Hom}_{\mathcal{L}(G)}(\Delta\mathbb{Z},\mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A}) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0)\\
&\to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I,\mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(KG^I,\mathcal{A}^0) \to \mathrm{Ext}^2_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0) \to \cdots
\end{aligned}$$

is exact.

Proof. We only need to show exactness at $\mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A})$ and $\mathrm{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I,\mathcal{A}^0)$ and conclude the result following the fact that exactness at other positions in the sequence is obtained by definition.

For exactness at $\mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A})$, we consider the sequence

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{L}(G)}(\Delta\mathbb{Z},\mathcal{A}) \to \mathrm{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G,\mathcal{A}) \xrightarrow{i} \mathrm{Hom}_{\mathcal{L}(G)}(KG,\mathcal{A})\\
&\xrightarrow{=} \mathrm{Hom}_{\mathcal{L}(G^I)}(KG^I,\mathcal{A}^0) \hookrightarrow \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z},\mathcal{A}^0)
\end{aligned}$$

We have that $\ker i = \operatorname{im}\big(\operatorname{Hom}_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) \to \operatorname{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})\big)$ by definition. Since δ is the composite of i, equality map and the injection, it follows that

$$\ker\delta = \ker i = \operatorname{im}\big(\operatorname{Hom}_{\mathcal{L}(G)}(\Delta\mathbb{Z}G, \mathcal{A}) \to \operatorname{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})\big)$$

and so the exactness at $\operatorname{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A})$ follows.
Now $\operatorname{im}\delta = \operatorname{im}\Big(\operatorname{Hom}_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \hookrightarrow \operatorname{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0)\Big)$ and so considering the sequence

$$\operatorname{Hom}_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \xrightarrow{\delta} \operatorname{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \operatorname{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0)$$

we have that $\ker\delta = \operatorname{im}\Big(\operatorname{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \operatorname{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0)\Big)$ and hence exactness at $\operatorname{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0)$ follows.
Therefore the given Hom –Ext sequence is exact as desired. $\square$

From Theorem 3.3.7 together with identifying the cohomology groups with the corresponding $\operatorname{Ext}^i_{-}(-,-)$ we have that δ is precisely the composite

$$H^n(E(G), \mathcal{A}) \to \operatorname{Ext}^n_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0)$$

and so the sequence

$$\begin{aligned} &0 \to H^0(\mathcal{L}(G), \mathcal{A}) \to H^0(E(G), \mathcal{A}) \xrightarrow{\delta} H^1(\mathcal{L}(G^I), \mathcal{A}^0) \\ &\to \operatorname{Ext}^1_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) \to \operatorname{Ext}^1_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to H^2(\mathcal{L}(G^I), \mathcal{A}^0) \to \cdots \end{aligned}$$

is exact. Now we present the following connections of cohomology groups which will be applied in the next section to obtain the main result of this chapter. The ideas motivated by Lemma 4.3 of [29] for inverse semigroups.

Theorem 3.4.4. *Suppose $\mathcal{A}$ is a G–module and let $\mathcal{A}^0$ and $\mathcal{A}^I$ be G^I-modules obtained from $\mathcal{A}$ by attaching the singleton group $\{0\}$ and $\varprojlim^{E(G)}\mathcal{A}$ at I in $E(G^I)$ respectively. Then*

1. the sequence

$$0 \to H^0(\mathcal{L}(G), \mathcal{A}) \to H^0(E(G), \mathcal{A}) \xrightarrow{\delta} H^1(\mathcal{L}(G^I), \mathcal{A}^0) \to H^1(\mathcal{L}(G^I), \mathcal{A}^I) \to 0$$

is exact,

2. $H^n(\mathcal{L}(G^I), \mathcal{A}^0) \cong H^n(\mathcal{L}(G^I), \mathcal{A}^I)$ *for* $n > 1$,

3. $H^n(\mathcal{L}(G^I), \mathcal{A}^0) \cong \mathrm{Ext}^{n-1}_{\mathcal{L}(G)}(KG, \mathcal{A})$ *for* $n > 0$.

Proof. Let $\mathcal{B}$ be a G^I–module. Recall that the G^I-map $\mathcal{B} \to \mathcal{A}^I$ is the family of maps $\varphi : \mathcal{B}_e \to \mathcal{A}_e$ for $e \in E(G)$ and the map $\mathcal{B}_I \to \varprojlim^{E(G)} \mathcal{A}$ uniquely determined by φ. Hence the map $\mathcal{A}^0 \to \mathcal{A}^I$ is uniquely determined by the map $\mathcal{A}^0_e \to \mathcal{A}^I_e$ for $e \in E(G)$. Let $\mathcal{A}^I/\mathcal{A}^0$ be the quotient G^I-module. The quotient module $\mathcal{A}^I/\mathcal{A}^0$ associates $I \in E(G^I)$ with the module $\varprojlim^{E(G)} \mathcal{A}$ and zeros elsewhere. Thus $\mathcal{A}^I/\mathcal{A}^0$ is in fact the result of the functor $\Theta : \mathbf{Ab} \to \mathrm{Mod}_{\mathcal{L}(G^I)}$ where Θ associates $\mathcal{B} \in \mathbf{Ab}$ the G^I–module defined by assigning the group $\mathcal{B}$ at $I \in E(G^I)$ and zeros elsewhere. It is clear that the quotient module is the G^I–module associated with the abelian group $\mathcal{B} = \varprojlim^{E(G)} \mathcal{A}$ by Θ since $\Theta(\mathcal{B}) = \mathcal{A}^I/\mathcal{A}_0$. We have that $\varprojlim^{\mathcal{L}(G)} \mathcal{A}^I/\mathcal{A}^0 = \varprojlim^{E(G)} \mathcal{A}$. The functor Θ is exact and preserves injectives since it is right adjoint to the evaluation functor which is exact and preserves injectives. By definition of the cohomology functor on ordered groupoids we get

$$H^n(\mathcal{L}(G^I), \mathcal{A}^I/\mathcal{A}^0) = \begin{cases} \varprojlim^{\mathcal{L}(G^I)} \mathcal{A}^I/\mathcal{A}^0 = \varprojlim^{E(G)} \mathcal{A} & n = 0 \\ 0 & \text{for } \mathcal{A} \text{ injective} \end{cases}$$

Consider the short exact sequence of G^I-modules $0 \to \mathcal{A}^0 \to \mathcal{A}^I \to \mathcal{A}^I/\mathcal{A}^0 \to 0$. Applying the covariant functor $\mathrm{Ext}^n_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, -)$ to the short exact sequence of the $\mathcal{L}(G^I)$-modules generates a long exact Hom-Ext sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \mathrm{Hom}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^I) \to \mathrm{Hom}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^I/\mathcal{A}^0) \\ &\to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^I) \to \mathrm{Ext}^1_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^I/\mathcal{A}^0) \to \cdots \end{aligned}$$

and via identifications with the cohomology groups with appropriate coefficients gives

$$
\begin{aligned}
&0 \to H^0(\mathcal{L}(G^I), \mathcal{A}^0) \to H^0(\mathcal{L}(G^I), \mathcal{A}^I) \to H^0(\mathcal{L}(G^I), \mathcal{A}^I/\mathcal{A}^0) \to H^1(\mathcal{L}(G^I), \mathcal{A}^0) \\
&\to H^1(\mathcal{L}(G^I), \mathcal{A}^I) \to H^1(\mathcal{L}(G^I), \mathcal{A}^I/\mathcal{A}^0) \to \cdots \\
&\to H^n(\mathcal{L}(G^I), \mathcal{A}^0) \to H^n(\mathcal{L}(G^I), \mathcal{A}^I) \to H^n(\mathcal{L}(G^I), \mathcal{A}^I/\mathcal{A}^0) \to \cdots .
\end{aligned}
$$

The result in (2) is obtained from the following. We have shown that for $n \geqslant 1$ the cohomology group $H^n(\mathcal{L}(G^I), \mathcal{A}^I/\mathcal{A}^0) = 0$ hence the isomorphism $H^n(\mathcal{L}(G^I), \mathcal{A}^0) \cong H^n(\mathcal{L}(G^I), \mathcal{A}^I)$ desired.

Now $H^0(\mathcal{L}(G^I), \mathcal{A}^I) = \mathrm{Ext}^0_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^I) = \mathrm{Hom}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^I)$. We get that every G^I–map $\Delta\mathbb{Z} \to \mathcal{A}^I$ corresponds to an G^I–map $\mathbb{Z}G^I \to \mathcal{A}^I$ by left composition with the augmentation map $\mathbb{Z}G^I \to \Delta\mathbb{Z}$. Thus $H^0(\mathcal{L}(G^I), \mathcal{A}^I) = \mathrm{Ext}^0_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^I)$. However the G^I–map $\mathbb{Z}G^I \to \mathcal{A}^I$ is determined by the restriction $\mathbb{Z}G \to \mathcal{A}$ hence we have that $\mathrm{Ext}^0_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^I) = \mathrm{Ext}^0_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) = H^0(E(G), \mathcal{A})$ from Theorem 3.3.7. Therefore $H^0(\mathcal{L}(G^I), \mathcal{A}^I) = H^0(E(G), \mathcal{A})$.

By definition, $H^0(\mathcal{L}(G^I), \mathcal{A}^0) = \mathrm{Ext}^0_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) = \mathrm{Hom}_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0)$. But G^I–maps $\Delta\mathbb{Z} \to \mathcal{A}^0$ are determined by the G–maps $\Delta\mathbb{Z} \to \mathcal{A}$ and so $H^0(\mathcal{L}(G^I), \mathcal{A}^0) = \mathrm{Ext}^0_{\mathcal{L}(G^I)}(\Delta\mathbb{Z}, \mathcal{A}^0) = \mathrm{Ext}^0_{\mathcal{L}(G)}(\Delta\mathbb{Z}, \mathcal{A}) = H^0(\mathcal{L}(G), \mathcal{A})$.

So the Hom-Ext sequence in lower dimensions now reads

$$
H^0(\mathcal{L}(G), \mathcal{A}) \to H^0(E(G), \mathcal{A}) \to H^0(\mathcal{L}(G^I), \mathcal{A}^I/\mathcal{A}^0) \to H^1(\mathcal{L}(G^I), \mathcal{A}^0) \to H^1(\mathcal{L}(G^I), \mathcal{A}^I) \to 0
$$

Using the map δ leads to the result in (1) as desired.

Now we obtain the results in (3) as follows. The short exact sequence $0 \to KG^I \to \mathbb{Z}G^I \to \Delta\mathbb{Z} \to 0$ of $\mathcal{L}(G^I)$ modules generates the long exact sequence

$$
\begin{aligned}
&\cdots \to H^n(\mathcal{L}(G^I), \mathcal{A}^0) \to H^n(E(G^I), \mathcal{A}^0) \\
&\to \mathrm{Ext}^n_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0) \to H^n(E(G^I), \mathcal{A}^0) \to \cdots
\end{aligned} \tag{3.4.4}
$$

by applying the contravariant functor $\mathrm{Ext}^i_{\mathcal{L}(G^I)}(-, \mathcal{A}^0)$. In particular for $n = 0$ $H^0(E(G^I), \mathcal{A}^0) = \mathrm{Ext}^0_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) = \mathrm{Hom}_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0)$. Now any morphism

$(\mathbb{Z}G^I)_I \xrightarrow{\psi_I} \mathcal{A}_I^0$ is determined by the maps $(\mathbb{Z}G^I)_I \xrightarrow{\lhd(I,e)} (\mathbb{Z}G^I)_e$ for some $e \in G_0$ and $(\mathbb{Z}G^I)_e \xrightarrow{\psi_e} \mathcal{A}_e^0$. Since the diagram

$$\begin{tikzcd}
(\mathbb{Z}G^I)_I \arrow[r, "\psi_I"] \arrow[d, "\lhd(I,e)"'] & \mathcal{A}_I^0 = 0 \arrow[d, "\lhd(I,e)"] \\
(\mathbb{Z}G^I)_e \arrow[r, "\psi_e"] & \mathcal{A}_e^0
\end{tikzcd}$$

commutes, suppose $e \in (\mathbb{Z}G^I)_e$, then $e\psi_e = 0 \in \mathcal{A}_e^0$. And so $\mathrm{Hom}_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0) = 0$. Therefore $H^n(\mathrm{Hom}_{\mathcal{L}(G^I)}(\mathbb{Z}G^I, \mathcal{A}^0)) = 0$ for $n \geqslant 1$. Therefore (3.4.4) modifies to

$$\cdots \to H^n(\mathcal{L}(G^I), \mathcal{A}^0) \to 0 \to \mathrm{Ext}^n_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0) \to 0 \quad (3.4.5)$$

for $n \geqslant 1$. The maps $\mathrm{Ext}^n_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) \to H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0)$ are then necessarily isomorphisms and using the identification $\mathrm{Ext}^n_{\mathcal{L}(G^I)}(KG^I, \mathcal{A}^0) = \mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A})$ from (3.4.3) we get the isomorphisms $\mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A}) \cong H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0)$ the desired result in (3). □

3.4.3 Main results

We dedicate this section to discussing the main results of this chapter. Our discussion follows from [29] on inverse semigroups. Loganathan in [29] relates the cohomology of an inverse semigroup with that of its semilattice of idempotents. We present an analogous result for ordered groupoids in the following theorem.

Theorem 3.4.5. *Let G be an ordered groupoid with set of identities $E(G)$. Suppose $\mathcal{A}$ is a G–module. Then the sequence*

$$\begin{aligned}
0 \quad \to \quad & H^0(\mathcal{L}(G), \mathcal{A}) \to H^0(E(G), \mathcal{A}) \xrightarrow{\delta} H^1(\mathcal{L}(G^I), \mathcal{A}^0) \to \\
\cdots \quad & H^{n-1}(E(G), \mathcal{A}) \xrightarrow{\delta} H^n(\mathcal{L}(G^I), \mathcal{A}^0) \to H^n(\mathcal{L}(G), \mathcal{A}) \to \cdots
\end{aligned}$$

of cohomology groups is exact.

Proof. Suppose G is an ordered groupoid. Consider the sequence

$$0 \to KG \to \mathbb{Z}G \to \Delta\mathbb{Z} \to 0$$

of G–modules. Applying the contravariant functor $\mathrm{Ext}^i_{\mathcal{L}(G)}(-, \mathcal{A})$ generates a long exact sequence

$$\begin{aligned} &H^0(\mathcal{L}(G), \mathcal{A}) \to \mathrm{Ext}^0_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \to \mathrm{Ext}^0_{\mathcal{L}(G)}(KG, \mathcal{A}) \to H^1(\mathcal{L}(G), \mathcal{A}) \\ &\to \mathrm{Ext}^1_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \to \mathrm{Ext}^1_{\mathcal{L}(G)}(KG, \mathcal{A}) \to H^2(\mathcal{L}(G), \mathcal{A}) \to \cdots \end{aligned} \tag{3.4.6}$$

By Theorem 3.3.7, $\mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) = H^n(E(G), \mathcal{A})$. We recall that δ is the composite

$$H^n(E(G), \mathcal{A}) = \mathrm{Ext}^n_{\mathcal{L}(G)}(\mathbb{Z}G, \mathcal{A}) \to \mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A}) \xrightarrow{\cong} \mathrm{Ext}^n_{\mathcal{L}(G)}(KG^I, \mathcal{A}^0) \to H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0)$$

By Theorem 3.4.4, $\mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A}) \cong H^{n+1}(\mathcal{L}(G^I), \mathcal{A}^0)$. Thus δ is the same as the map $H^n(E(G), \mathcal{A}) \to \mathrm{Ext}^n_{\mathcal{L}(G)}(KG, \mathcal{A})$. Therefore making the identifications from theorem 3.3.7 and theorem 3.4.4 together with the map δ gives the exact sequence

$$\begin{aligned} 0 \;\to\; & H^0(\mathcal{L}(G), \mathcal{A}) \to H^0(E(G), \mathcal{A}) \xrightarrow{\delta} H^1(\mathcal{L}(G^I), \mathcal{A}^0) \to \\ \cdots\; & H^{n-1}(E(G), \mathcal{A}) \xrightarrow{\delta} H^n(\mathcal{L}(G^I), \mathcal{A}^0) \to H^n(\mathcal{L}(G), \mathcal{A}) \to \cdots \end{aligned}$$

as desired. $\square$

Chapter 4

Homology of Level Groupoids

The main result, Theorem 4.5.2 of this Chapter, describes the connection between the homology groups of some class of ordered groupoids and that of their *level* groupoids. Given an ordered groupoid G, the construction of the level groupoid $G_{\updownarrow}$ from G first appeared in [16] where Gilbert used $G_{\updownarrow}$ as a tool to investigate the structural properties of ordered groupoids that extend the idea of the P–theorem for inverse semigroups (see details in [36]). The goal of this chapter is inspired by the results due to Loganathan on inverse semigroups in [29, section 3]. Our result recovers the connection between the homology groups of inverse semigroups and that of their associated maximum group homomorphic image established by Loganathan.

The major contribution is captured in the following theorem:

Theorem 4.5.2 . *For any β–transitive groupoid G and G–module $\mathcal{A}$, and any $n \geqslant 0$, the homology groups $H_n(G, \mathcal{A})$ and $H_n(G_{\updownarrow}, \varinjlim^{E(G)} \mathcal{A})$ are isomorphic.*

We will explain β–transitivity and the levelling construction in detail later in this chapter. The discussion of the results in Theorem 4.5.2 is captured in the five sections of this chapter. We commence with the construction of the level groupoid. The second section deals with the notion of the β congruence on ordered groupoids: a generalisation of the minimum group congruence on inverse semigroups. In the third section, we discuss the relationship between the levelling and β relations. The fourth section introduces the idea of modules over the associated category of $G_{\updownarrow}$. We discuss the correspondence of the module categories of the level groupoid and β–transitive class of ordered groupoids. The final section present the details of the main result.

4.1 Level groupoid

In this section we present the construction of level groupoids from ordered groupoids. The content of our discussion is derived from [16]. The preliminary data of the construction is the notion of *universal* groupoids presented by Higgins in [21]. We spend the next paragraph to explain this concept.

4.1.1 Universal groupoid

The task of constructing universal groupoids involves identifying elements of groupoids in such a way that the resulting structure is a groupoid and possesses some universal property. This is analogous to construction of terminal and initial objects in a category. The universal groupoid $\mathcal{U}_\sigma(G)$ is precisely a pushout of the groupoid maps $G_0 \to G$ and $\sigma : G_0 \to V$ where V is a set considered as an ordered groupoid and so $G \xrightarrow{\sigma'} \mathcal{U}_\sigma(G)$ is a universal map, that is for every such universal groupoid–map $G \xrightarrow{f} H$ over σ, there is a unique groupoid map $\mathcal{U}_\sigma(G) \xrightarrow{f'} H$ such that $\sigma' f' = f$. The following paragraph is dedicated to explaining the formal details.

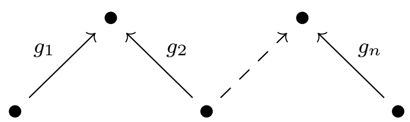
Let G be an groupoid with object set G_0, and some function $\sigma : G_0 \to V$ where V is a set. We define a graph G^σ of G induced by σ as follows.

Definition Let $\sigma : G_0 \to V$ be a function of sets. The graph G^σ is defined as having vertex set V and edges the non-identity morphisms in G. The source and target maps are given by $(g)\mathbf{d}^\sigma = [(g)\mathbf{d}]\sigma$ and $(g)\mathbf{r}^\sigma = [(g)\mathbf{r}]\sigma$ respectively. It is easy to see that σ and the identity map on morphisms induces a graph map $\Gamma(G) \to G^\sigma$ where $\Gamma(G)$ is the underlying directed graph of G.

A path p in G^σ is a sequence of edges that join up in sequence and $\ell(p)$ is the length of p, defined as the number of edges in p. Paths in G^σ together with concatenation of paths as composition forms the *category of directed paths* in G^σ which we denote by $\vec{P}(G^\sigma)$. The composite map

$$G \longrightarrow G^\sigma \longrightarrow \vec{P}(G^\sigma)$$

defines a map $G \to \vec{P}(G^\sigma)$ whose restriction to the set of identities is $\sigma : E(G) \to E(\vec{P}(G^\sigma))$. It is understood that composable morphisms say $g_1 \circ g_2 = g$ in G corresponds to a path of length 2 in $\vec{P}(G^\sigma)$ and so cannot be the image of the morphism g which corresponds to a path of length 1. We thus modify the definition of paths by using *elementary reduction* so that $g_1 \circ g_2$ and g correspond to the same path. Let $p = a_1 \cdots a_n$ be a path in $\vec{P}(G^\sigma)$. An elementary reduction of p is the substitution of adjacent edges obtained from composable morphisms $g_i \circ g_{i+1} = g_k$ with the edge associated with g_k and the deletion of edges corresponding to identity morphisms. A path that admits no elementary reduction is called *σ-reduced* path. So in a reduced path $p = a_1, \cdots, a_n$ no two adjacent edges are the images of composable morphisms in G. A typical reduced path consist of sequence of edges that are images of morphisms of the form



in G. The elementary reduction defines an equivalence relation on the path category $\vec{P}(G^\sigma)$ of G^σ. Two paths are related if one can be obtained from the other via elementary reduction. Equivalent paths have the same source and target. Multiplication of equivalent classes is defined by the concatenation of reduced paths $[p][q] = [pq]$ where pq is a well defined reduced word and the class $[()_e]$ where $()_e$ is the empty path at the identity e is the identity class at the object e. Suppose $p = a_1, \cdots, a_n$ is a reduced path from e to f then we set the inverse path to be the sequence $\bar{p} = a_n^{-1}, a_{n-1}^{-1}, \cdots, a_1^{-1}$ of edges from f to e corresponding to the inverses of g_i's in p. So the we say $[\bar{p}]$ is the inverse of $[p]$. Therefore the path category modulo the equivalence relation is a groupoid, denoted by $\mathcal{U}_\sigma(G)$. Then we get the map $G \xrightarrow{\sigma'} \mathcal{U}_\sigma(G)$ defined by $g \mapsto [g]$ and so considering G_0 and V as trivial groupoids, the diagram of groupoids

$$\begin{tikzcd} G_0 \arrow[r, "\sigma"] \arrow[d, hook] & V \arrow[d, hook] \\ G \arrow[r, "\sigma'"] & \mathcal{U}_\sigma(G) \end{tikzcd}$$

commutes. The diagram is a pushout square in the category of groupoids and so is called the *universal groupoid*. The universality of $\mathcal{U}_\sigma(G)$ is expressed in the following. proposition.

Proposition 4.1.1. ([21, Proposition 19′])

1. *Suppose G is a groupoid and $\sigma : G_0 \to V$ is some function of sets. Then there is a universal morphism $G \to \mathcal{U}_\sigma(G)$ with the restriction morphism $\sigma : E(G) \to E(\mathcal{U}_\sigma(G))$,*

2. *$\mathcal{U}_\sigma$ is a functor from the G_0–groupoid to the V–groupoid uniquely determined up to isomorphism by σ.*

Proof. See [21] for the detailed proof. □

4.1.2 Level groupoid

A functor $\zeta : G \to H$ between ordered groupoids is *levelling* if $g \leqslant g'$ implies that $g\zeta = g'\zeta$. Now suppose G is an ordered groupoid. Define $\updownarrow$ as the smallest equivalence relation on G generated by the partial order $\leqslant$. We say $g \updownarrow h$ if there is a sequence of elements $g_1, \cdots, g_n$ with $g_1 = g$ and $g_n = h$ and either $g_i \geqslant g_{i+1}$ or vice versa. Take the restriction of $\updownarrow$ to the set of identities of G. The restriction generates a set of equivalence classes of identities. Denote by $\lambda' : G_0 \to G_0/\updownarrow$ the function of sets. We construct the universal groupoid $\mathcal{U}_{\lambda'}(G)$ via the description in the preceding subsection. Recall from chapter 1 that the pseudoproduct is a categorical extension of products in ordered groupoids. The pseudoproduct coincides with the usual groupoid product in the unordered case. When G is ordered, the set of composable morphisms can be much bigger. So suppose $x, y \in G$ and that the pseudoproduct $x * y$ is defined. Then $x\mathbf{d} = x^{-1}\mathbf{r} \updownarrow (x * y)\mathbf{d}$ and $y\mathbf{r} \updownarrow (x * y)\mathbf{r}$. Thus $y^{-1}x^{-1} \cdot (x * y)$ is an element of the local group at $(y\mathbf{r})\lambda'$ in $\mathcal{U}_{\lambda'}(G)$. Let $\mathcal{N}$ be the normal subgroupoid of $\mathcal{U}_{\lambda'}(G)$ generated by elements of the form $y^{-1}x^{-1} \cdot (x * y)$. We define $G_{\updownarrow}$ as the quotient groupoid $\mathcal{U}_{\lambda'}(G)/\mathcal{N}$. The groupoid $G_{\updownarrow}$ is called the *level* groupoid of G with the quotient map

$$\lambda : G \to \mathcal{U}_{\lambda'}(G)/\mathcal{N}.$$

Lemma 4.1.2. ([16, Lemma 2.1]) *Let G be an ordered groupoid. Then $\lambda : G \to G_{\updownarrow}$ is a levelling functor, and given that the morphism of groupoids $\theta : G \to H$ is a levelling functor, then there exist a unique functor $\theta^* : G_{\updownarrow} \to H$ such that $\theta = \lambda\theta^*$.*

Proof. Let G be an ordered groupoid. The first part of the proof is mainly to show that for $x, y \in G$ and $x \leqslant y$ then $x\lambda = y\lambda$. Given that $x \leqslant y$ implies $xx^{-1} \leqslant yy^{-1}$. Thus the pseudoproduct $x^{-1} * y$ exist in G and is defined by

$$x^{-1} * y = x^{-1}(y|xx^{-1}) = x^{-1}x$$

It follows that $(x^{-1} * y)\lambda = (x^{-1}x)\lambda$ in $G_{\updownarrow}$. Therefore $y\lambda = x\lambda$ and so λ is a levelling functor.

Now it suffices to show that θ induces a unique functor $G_{\updownarrow} \to H$ which is necessarily levelling. Suppose $e, f \in G_0$ and $e \updownarrow f$. Given that θ is levelling implies $e\theta = f\theta$ in H hence θ induces $\theta' : G_{\updownarrow} \to H$. Also, if the pseudoproduct $x * y$ is difined in G. Then there exist some $z \in G_0$ such that $z \leqslant x^{-1}x,\ yy^{-1}$ so that $x * y = (z|x)(y|z)$. Thus

$$(x * y)\theta' = ((z|x)(y|z))\theta = x\theta y\theta = x\theta' y\theta'$$

since θ is levelling. Hence θ induces the unique functor $\theta^* : G_{\updownarrow} \to H$ given by $x\lambda = x\theta$ on $G_{\updownarrow}$. $\square$

Below are some examples of level groupoids which arise frequently in studies.

Example 4.1.1 Let G be an inductive groupoid. The set G_0 is a meet semilattice and so all elements of G_0 are $\updownarrow$ related. Hence the restriction of $\updownarrow$ to G_0 yields a singleton set, that is $G_0/\updownarrow = \{*\}$. It follows that the universal groupoid is a group and hence the level groupoid $G_{\updownarrow}$ is also a group. This is the maximum group homomorphic image of the associated inverse semigroup widely known (see [37] and [35]). The connexion between the level groupoid of inductive groupoids and the maximum group homomorphic image of the associated inverse semigroups is presented in details in section 5.3 of this thesis.

Example 4.1.2 Let E be a semilattice and F a covariant functor $F : E \to \mathbf{Grp}$ from E to the category of groups. So that F assigns groups G_e to $e \in E$ such that $G_e \neq G_f$ if $e \neq f$ in E. Let ϕ^e_f be the unique morphism $G_e \to G_f$ whenever $e \geqslant f$ in E. The disjoint union of groups $\bigcup_{e\in E} G_e$ is an ordered groupoid G with ordering determined by ϕ. In particular, $a \geqslant b$ given that $a\phi^e_f = b$ for $a \in G_e$ and $b \in G_f$ and $e \geqslant f$ in E. The level groupoid is precisely $\varinjlim^E G$.

Example 4.1.3 Let G_1 and G_2 be groups and $f : G_1 \to G_2$ a group homomorphism. Let G be the glued ordered groupoid $G = G_1 \sqcup_f G_2$ as defined in Chapter 2. Since $e_1 f = e_2$ we have that $e_2 \leqslant e_1$ and so $e_2 \updownarrow e_1$. It follows that $G_0/\updownarrow$ is the singleton set and thus $\lambda : G_{\updownarrow} \to G_0/\updownarrow$ is defined by $e_i\lambda = e$. And so $U_\lambda(G)$ is the free product $G_1 * G_2$. Precisely, elements of the universal groupoid are strings of alternating terms from G_1 and G_2. Now suppose the pseudoproduct $x * y$ exist in G. Then we have that for $x \in G_1$ and $y \in G_2$ then $y \leqslant x$ so that $e_2 \leqslant [(x)f]\mathbf{r}, y\mathbf{d}$ and so

$$x * y = (e_2|(x)f) \cdot (y|e_2) = (x)f \cdot y \, .$$

Define N as the smallest subgroup of $U_\lambda(G)$ containing elements $x \cdot y \cdot (x * y)^{-1}$ for $x, y \in \{G_1,\ G_2\}$. Suppose $x, y \in G_i$ then the elements of N are the identity elements in G_i. Let $x \in G_1$ and $y \in G_2$. Then the elements of N are generated by $x \cdot y \cdot ((x)f \cdot y)^{-1} = x \cdot ((x)f)^{-1}$. So N is simply the smallest normal subgroup of $U_\lambda(G)$ containing $x \cdot ((x)f)^{-1}$ for $x \in G_1$. $G_{\updownarrow} = U_\lambda(G)/N \cong G_2$ is a group. That is we identify x with its image $(x)f$. This is an embedding of G_1 into G_2 when f is injective.

Let G be a presheaf of groups over poset. Suppose the underlying graph is a Dynkin graph of type A_n and the order in G is determined by the monotonic increasing map in A_n. Then the level groupoid $G_{\updownarrow} \cong G_n$ and can be interpreted as an embedding into G_n if the morphisms in G are injective.

Example 4.1.4 The action of a group H on a poset P yields an ordered groupoid G. We define morphisms of G by (e, h) with $e \in P$ and $h \in H$ together with the composition $(e, h)(e', h') = (e, hh')$ whenever $e = e'$. The order in G is induced by partial order of P so that we have $(e, h) \leqslant (e', h')$ whenever $e \leqslant e'$ in P and $h = h'$ in H. It is straightforward to check that G is indeed an ordered groupoid. The level groupoid $\mathcal{U}_{\lambda}(G)$ has vertex set indexed by $\updownarrow$–classes in P. Hence the level groupoid $G_{\updownarrow} = \sqcup_{[e]\in P/\updownarrow} H_{[e]}$: the disjoint union of copies of H indexed by $\updownarrow$–classes in P.

4.2 β–transitive ordered groupoids

In [37], Munn tackles the problem of those congruences on semigroups which yield quotient semigroups with some predefined properties. He showed in his Corollary 2.9 that given an inverse semigroup S and a congruence σ on S defined by

$$x \; \sigma \; y \iff \text{there exist some } z \in S \text{ such that } z \leqslant x, \; y$$

then the quotient S/σ is a group and any congruence τ on S such that S/τ is a group implies $\tau \subset \sigma$. The quotient group S/σ is the maximal group homomorphic image of S. Gomes and Howie make use of the results of [37] in [18] to study the P–theorem for inverse semigroups with zero. The parallel result of [18] has been studied for ordered groupoids by Gilbert in [16]. Inspired by section 3 of [29] by Loganathan. Suppose C is a category with *nerve* NC. The *classifying space* of C is the *geometric realization* BC of NC. A functor $C \xrightarrow{\Gamma} C'$ of categories is called a *homotopy equivalence* if it induces a homotopy equivalence of classifying spaces. See [40] for details. In [29], Loganathan showed that the projection $S \to S/\sigma$ induces a contracting homotopy $\mathcal{L}(S) \to \mathcal{L}(S/\sigma)$. We dedicate this section to the study of the corresponding σ relation on ordered groupoids. The extension of the minimal group congruence on S to ordered groupoids appeared in [16] briefly with the name β as in [18] which we shall adopt for consistency.

Let G be an ordered groupoid. The relation β on G is defined by

$$g \, \beta \, h \iff \text{there exists } c \in G \text{ with } c \leqslant g \text{ and } c \leqslant h \, .$$

The definition of β is in fact the same as that of σ. So for G an inductive groupoid β coincides with the corresponding minimum group congruence σ on $G(S)$ induced by σ on S. It is evident that β is reflexive and symmetric. However transitivity of β is not always true as we can find a counter example.

Example 4.2.5 Consider the poset $Y = e, f, z, l, h$ with the order relations $e \geqslant z \leqslant f \geqslant l \leqslant h$. Let G be a presheaf of groups on Y and the ordering in G be determined by the morphisms $\phi_z^e : G_e \to G_z$, $\phi_z^f : G_f \to G_z$, $\phi_l^f : G_f \to G_l$ and $\phi_l^h : G_h \to G_l$.

$$\begin{tikzcd}
G_e \arrow[dr, dashed, "\phi_z^e"] & & G_f \arrow[dl, dashed, "\phi_z^f"'] \arrow[dr, dashed, "\phi_l^f"] & & G_h \arrow[dl, dashed, "\phi_l^h"'] \\
& G_z & & G_l &
\end{tikzcd}$$

It is clear that $a\beta b$ and $b\beta c$ however a and c are not β related for $a \in G_e$, $b \in G_f$ and $c \in G_h$.

Lemma 4.2.1. [16, section 2.2] *G is β–transitive if and only if every principal order ideal in G is a directed set.*

Now suppose β defined on G is a transitive relation then we say G is a *β–transitive* ordered groupoid. Every principal order ideal of inductive groupoids is a meet semilattice. As a result the relation β on inductive groupoids is transitive and so every inductive groupoid is *β–transitive*. If the poset of principal order ideals is a meet semilattice, then the ordered groupoid is said to be *principally inductive*.

We make the following proposition from ([26, Theorem 20])

Proposition 4.2.2. *Let G be a principally inductive groupoid. Then*

1. *the quotient groupoid G/β is an unordered groupoid,*

2. *given the ordered functor $\theta : G \to H$ into an unordered groupoid H, the relation $\beta \subset \ker\theta$*

Proof. 1. The result is a direct consequence of the definition

2. Let $z \leqslant a,\ b$ in G so that $a\beta b$. Since θ is an ordered functor hence $z\theta \leqslant a\theta,\ b\theta$. However H is unordered hence $z\theta = a\theta = b\theta$ and so $\beta \subset \ker\theta$

□

It is clear that if G is β–transitive then so is its poset of identities $E(G)$. However, the converse is false. Here we present a supporting argument. Let A and B be groups with a common subgroup C and let $i : C \hookrightarrow A$ and $j : C \hookrightarrow B$ be the inclusions. Consider the semilattice $\{0, e, f, 1\}$ with e, f incomparable, and define a semilattice of groups G by $G_1 = C, G_e = A, G_f = B$ and $G_0 = A \times B$ and with the obvious structure maps.

$$\begin{tikzcd} & G_0 & \\ G_e \arrow[ur, dashed] & & G_f \arrow[ul, dashed] \\ & G_1 \arrow[ul, dashed] \arrow[ur, dashed] & \end{tikzcd}$$

Then $ci\,\beta\,c\,\beta\,cj$ for all $c \in C$, but ci and cj are not β–related.

4.3 Corresespondence of levelling and β relations

Recall that $\updownarrow$ is an equivalence relation on G generated by the partial order $\leqslant$. It is clear that, as relations on G, we have $\beta\ \subseteq \updownarrow$. Now if $g \leqslant h$ then $g\,\beta\,h$, and so if β is transtive it is an equivalence relation containing $\leqslant$, and so $\updownarrow \subseteq\ \beta$. Hence in a β–transitive ordered groupoid, the relations $\updownarrow$ and β are the same. We shall denote the β–class of $g \in G$ by either $[g]$ or $g\lambda$.

A functor $\zeta : G \to H$ between ordered groupoids is *levelling* if $g \leqslant g'$ implies that $g\zeta = g'\zeta$ as discussed earlier. Each levelling functor factors through the universal levelling functor $\lambda : G \to G_{\updownarrow}$. The *level groupoid* $G_{\updownarrow}$ is defined to be the quotient of the universal groupoid $U_{\lambda}(G)$ on the quotient function $\lambda : E(G) \to E(G)/\updownarrow$, which coincides with $G \to G/\beta, e \mapsto [e]$ when G is β–transitive. It is clear that $G \to G/\beta$ is a levelling functor for β–transitive groupoid G. $G_{\updownarrow}$ is then the quotient of $U_{\lambda}(G)$ by the normal subgroupoid generated by all elements

$$y^{-1}x^{-1} \cdot (x * y)$$

defined whenever $x\mathbf{r}$ and $y\mathbf{d}$ are β related. That is there exist a greatest lower bound $i \in E(G)$ of $x\mathbf{r}$ and $y\mathbf{d}$, and $x * y = (i|x)(y|i) \in G$.

Proposition 4.3.1. ([16, Proposition 2.2]) *Let G be a β–transitive ordered groupoid. Then the quotient map $G \to G/\beta$ is the universal levelling functor $G \to G_{\updownarrow}$.*

An immediate observation from the above proposition is the widely known result that every group homomorphic image of an inverse semigroup S factors through the maximum group homomorphic image of S.

4.4 Functors on modules

This section presents the data of the associated category of the level groupoid and studies the corresponding module category. It turns out that in the case of inverse semigroups there is a homotopy equivalence between the associated categories of the maximum group homomorphic image and that of the inverse semigroup as investigated by Loganathan in [29, Theorem 3.3] and hence $\pi_1(\mathcal{L}(S), e) = S_{\updownarrow}$ for $e \in E(S)$. He then shows that the functor $\mathrm{Mod}_{S_{\updownarrow}} \to \mathrm{Mod}_{\mathcal{L}(S)}$ induced by the projection map $S \to S_{\updownarrow}$ has a left adjoint. This essentially leads to the existence of an isomorphism in the homology groups of $\mathcal{L}(S)$ and $S_{\updownarrow}$. The results of Loganathan translates to the corresponding inductive groupoid. In this section we extend the results on morphisms of the module categories beyond the inductive case by Loganathan to β-transitive groupoids which has inductive groupoids as a subclass.

4.4.1 Morphisms of modules

Let G be an ordered groupoid. It is understood that there is no ordering prescribed for the level groupoid $G_{\updownarrow}$ and so essentially $\mathcal{L}(G_{\updownarrow}) = G_{\updownarrow}$. A $G_{\updownarrow}$-module $\mathcal{B}$ is a functor $G_{\updownarrow} \to \mathbf{Ab}$ associating each $[e]$ for $e \in G_0$ the abelian group $\mathcal{B}_{e\lambda}$. The action of $G_{\updownarrow}$ on $\mathcal{B}$ is given as follows. Let $g\lambda \in G_{\updownarrow}([e],[f])$ then the action of g on $a \in \mathcal{B}_{e\lambda}$ is given by $a \lhd (g\lambda) \in \mathcal{B}_{f\lambda}$. Morphisms of $G_{\updownarrow}$–modules are natural transformations. $G_{\updownarrow}$–modules together with their corresponding morphisms constitute the functor category $\mathrm{Mod}_{G_{\updownarrow}}$. If $\mathcal{B}$ is a $G_{\updownarrow}$–module then we can *inflate* $\mathcal{B}$ to obtain a G–module $\mathcal{B}^{\leqslant}$ as follows:

- for $e \in E(G)$ we have $\mathcal{B}^{\leqslant}_e = \mathcal{B}_{e\lambda}$,
- if $e \geqslant f$ then $e\lambda = f\lambda$ and $\alpha^e_f = \mathrm{id}$,
- for $x, y \in E(G)$ and for each $g \in G(x,y)$, the map $\lhd g : \mathcal{B}_{x\lambda} \to \mathcal{B}_{y\lambda}$ is the action of $g\lambda$.

The data above defines a morphism $\mathrm{Mod}_{G_{\updownarrow}} \to \mathrm{Mod}_{\mathcal{L}(G)}$ of the functor categories since, if $\xi : \mathcal{B} \to \mathcal{B}'$ is a $G_{\updownarrow}$-map then we obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{B}_{e\lambda} \arrow[rr, "\xi_{e\lambda}"] \arrow[d, equal] & & \mathcal{B}'_{e\lambda} \arrow[d, equal] \arrow[ddr, bend left, "\lhd(e,g)"] & \\
\mathcal{B}_{(gg^{-1})\lambda} \arrow[rr, "\xi_{(gg^{-1})\lambda}"] \arrow[dr, "\lhd g\lambda"] & & \mathcal{B}'_{(gg^{-1})\lambda} \arrow[dr, "\lhd g\lambda"] & \\
& \mathcal{B}_{(g^{-1}g)\lambda} \arrow[rr, "\xi_{(gg^{-1})\lambda}"] & & \mathcal{B}'_{(g^{-1}g)\lambda}
\end{tikzcd}$$

and so generating the G-map $\xi^{\leqslant} : \mathcal{B}^{\leqslant} \to (\mathcal{B}')^{\leqslant}$ with $\xi^{\leqslant}_e = \xi_{e\lambda}$. We shall refer to the functor defined as the *inflation* functor.

Lemma 4.4.1. *The inflation functor* $\mathrm{Mod}_{G_{\updownarrow}} \to \mathrm{Mod}_{\mathcal{L}(G)}$ *preserves epimorphisms.*

Proof. An epimorphism in the category of modules is by definition a family of surjections. Thus if ξ is a surjection in $\mathrm{Mod}_{G_{\updownarrow}}$ then it follows that $\xi^{\leqslant}$ is a surjection in $\mathrm{Mod}_{\mathcal{L}(G)}$. $\square$

Suppose G is inductive, then the inflation functor coincides with the module morphism $\mathcal{L}(S) \to \mathcal{L}(S/\sigma)$ which is in fact a homotopy equivalence for the associated inverse semigroups. The inclusion $E(S) \to \mathcal{L}(S)$ determines a restriction functor $\mathrm{Mod}_{\mathcal{L}(S)} \to \mathrm{Mod}_{E(S)}$ which has an adjoint pair. For the parallel discussion we devote the next paragraph to discussing a morphism $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathrm{Mod}_{G_\updownarrow}$ for β–transitive ordered groupoids G and later show that it is left adjoint to the inflation.

Let $\mathcal{B}^{\leqslant}$ be the inflation of the $G_\updownarrow$-module $\mathcal{B}$. Suppose that $\mathcal{A}$ is an G–module and that we are given a map $\phi : \mathcal{A} \to \mathcal{B}^{\leqslant}$, with component maps $\phi_e : \mathcal{A}_e \to \mathcal{B}_{e\lambda}, (e \in E(G))$. Whenever $e \geqslant f$ we obtain a commutative triangle

$$\begin{tikzcd}
\mathcal{A}_e \arrow[rr, "\alpha_f^e"] \arrow[dr, "\phi_e"'] & & \mathcal{A}_f \arrow[dl, "\phi_f"] \\
& \mathcal{B}_{e\lambda} &
\end{tikzcd}$$

(in which $\mathcal{B}_{e\lambda} = \mathcal{B}_{f\lambda}$). By the universal property of the colimit, $\varinjlim$, ϕ factors through $\varinjlim$ of $\mathcal{A}$ and so ϕ_e induce a map

$$\psi : \varinjlim^{E(G)} \mathcal{A} \to \mathcal{B}$$

and so if $\alpha_e : \mathcal{A}_e \to \varinjlim^{E(G)} \mathcal{A}$ is the canonical map, then we decompose the diagram into

$$\begin{tikzcd}
\mathcal{A}_e \arrow[dr, "\alpha_e"] \arrow[dd, "\alpha_f^e"'] \arrow[drr, bend left, "\phi_e"] & & \\
& \varinjlim^{E(G)} \mathcal{A} \arrow[r, "\psi"] & \mathcal{B}_{e\lambda} \\
\mathcal{A}_f \arrow[ur, "\alpha f"] \arrow[urr, bend right, "\phi_f"'] & &
\end{tikzcd}$$

so that $\phi_e = \alpha_e \psi$. Thus the map ψ determines the family of maps (ϕ_e). We will show that the map ψ is a $G_\updownarrow$–map for β–transitive G. We present the results in the following proposition.

Proposition 4.4.2. *Suppose G is β–transitive ordered groupoid and let $\mathcal{A}$ be a G–*

module then $\varinjlim^{E(G)}\mathcal{A}$ *is a* $G_{\updownarrow}$*–module.*

Proof. Let $\mathfrak{L} = \varinjlim^{E(G)}\mathcal{A}$ and consider the canonical maps $\alpha_e : \mathcal{A}_e \to \mathfrak{L}_{[e]}$ and $\lambda : G \to G_{\updownarrow}$. Suppose that $\overline{a} \in \mathfrak{L}_{[e]}$ with $\overline{a} = a\alpha_e$ for some $a \in \mathcal{A}_e$, and $g \in G$ with $gg^{-1}\,\beta\, e$. Then gg^{-1} and e have a lower bound ℓ in $E(G)$, and we define an action of $g\lambda$ on $\overline{a}$ by

$$\overline{a} \lhd g\lambda = (a\alpha^e_\ell \lhd (g|\ell))\alpha_z \tag{4.4.1}$$

where $z = (g|\ell)\mathbf{r}$. We have to check that this definition is independent of the choices made for ℓ, a and g.

If we choose a different lower bound ℓ' of gg^{-1} and e, then ℓ and ℓ' are β related hence have a lower bound ℓ'' by transitivity of β, and so it is sufficient to show, for independence from the choice of ℓ, that the outcome of (4.4.1) is unchanged by descent in the partial order, in the following sense.

Suppose that $a \in \mathcal{A}_e$, $gg^{-1} = e$ and that there is $f \leqslant e$. Let $y = g^{-1}g$ and $z = (g|f)\mathbf{r}$. Then (4.4.1) gives $\overline{a} \lhd g\lambda = a\alpha^e_e \lhd g\lambda = (a \lhd g)\lambda_y$ when computing at e. Now if we base the calculation at f we obtain $\overline{a} \lhd g\lambda = (a\alpha^e_f \lhd (g|f))\lambda_z$. But in $\mathfrak{L}(G)$,

$$(e, f)(f, (g|f)) = (e, (g|f)) = (e, (g|e))(y, z)$$

and so $a\alpha^e_f \lhd (g|f) = (a \lhd g)\alpha^y_z$. Hence

$$(a\alpha^e_f \lhd (g|f))\lambda_z = (a \lhd g)\alpha^y_z)\lambda_z = (a \lhd g)\lambda_y\,.$$

Therefore the definition is independent of the choice of ℓ.

We now consider the choice of a preimage for $\overline{a}$. Suppose that $a\alpha_e = b\alpha_x$. Then e and x have a lower bound u with $\overline{a} = a\alpha^e_u\alpha_u = b\alpha^x_u\alpha_u$. So again it suffices to check what happens if we apply (4.4.1) at u. We have

$$\begin{aligned}\overline{a} \lhd g\lambda &= (a \lhd g)\alpha_y \\ &= (a\alpha^e_u) \lhd (g|u))\alpha_z\end{aligned}$$

where now $z = (g|u)\mathbf{r}$. But as before, $a\alpha_u^e \lhd (g|u) = (a \lhd g)\alpha_z^y$ and $\alpha_z^y \alpha_z = \alpha_y$. Hence the definition in (4.4.1) is independent of the choice of a.

Finally, suppose that $g\lambda = h\lambda$. Then $gg^{-1} \,\beta\, hh^{-1}$ and so gg^{-1} and hh^{-1} have a lower bound $v \in E(G)$. Then

$$g\lambda = (g|v)\lambda = (h|v)\lambda = h\lambda\,.$$

Then acting with $(g|v)$ in (4.4.1) we obtain

$$\begin{aligned}\overline{a} \lhd (g|v)\lambda &= (a\alpha_v^e \lhd (g|v))\alpha_z \\ &= (a \lhd g)\alpha_z^y \alpha_z \\ &= (a \lhd g)\alpha_y\,.\end{aligned}$$

Thus the definition in (4.4.1) is independent of the choice g. Therefore $\varinjlim^{E(G)}\mathcal{A}$ is a $G_{\updownarrow}$–module. □

Thus for a β–transitive ordered groupoid G, since ψ determines the family of maps (ϕ_e), we get that $(\phi_e) \mapsto \psi$ is an injection

$$\rho : \mathrm{Mod}_{\mathcal{L}(G)}(\mathcal{A}, \mathcal{B}^{\leqslant}) \to \mathrm{Mod}_{G_{\updownarrow}}(\varinjlim^{E(G)}\mathcal{A}, \mathcal{B})\,. \tag{4.4.2}$$

Recall that from Loganathan's work on inverse semigroups, the injection for the corresponding inductive groupoid is necessarily an isomorphism so that $\varinjlim$ is the left adjoint of the inflation functor. We shall extend the results of Loganathan by showing that if G is β–transitive then the injection is in fact a natural bijection and so the inflation is right adjoint to the colimit functor. We put the result in the following theorem.

Theorem 4.4.3. *Let G be a β–transitive ordered groupoid. Then the functor $\varinjlim^{E(G)}$ from $\mathrm{Mod}_{\mathcal{L}(G)}$ to $\mathrm{Mod}_{G_{\updownarrow}}$ is left adjoint to the inflation functor and hence is right exact and preserve epimorphisms.*

Proof. Let $\mathcal{A} \in \mathrm{Mod}_{\mathcal{L}(G)}$ and $\mathcal{B} \in \mathrm{Mod}_{G_{\updownarrow}}$. We have already defined the morphism $\rho : \mathrm{Mod}_{\mathcal{L}(G)}(\mathcal{A}, \mathcal{B}^{\leqslant}) \to \mathrm{Mod}_{G_{\updownarrow}}(\varinjlim^{E(G)}\mathcal{A}, \mathcal{B})$ which is in fact an injection. The aim of the rest of proof is to construct a morphism

$$\sigma : \mathrm{Mod}_{G_{\updownarrow}}(\mathfrak{L}, \mathcal{B}) \to \mathrm{Mod}_{\mathcal{L}(G)}(\mathcal{A}, \mathcal{B}^{\leqslant}) \tag{4.4.3}$$

where $\mathfrak{L} = \varinjlim^{E(G)} \mathcal{A}$ so that σ and ρ are inverses of each other.

For $[e] \in E(G)/\beta$ and $\psi : \mathfrak{L} \to \mathcal{B}$, consider the composition

$$\mathcal{A}_e \xrightarrow{\alpha_e} \mathfrak{L}_{[e]} \xrightarrow{\psi_{[e]}} \mathcal{B}_{[e]} = (\mathcal{B}^{\leqslant})_e \,.$$

We claim that this composition is an G–map. Now

$$\begin{aligned}(a \lhd (e,g))\alpha_{g^{-1}g}\psi_{[g^{-1}g]} &= (a\alpha^{e}_{gg^{-1}} \lhd g)\alpha_{g^{-1}g}\psi_{[g^{-1}g]} \\ &= (\overline{g} \lhd g\lambda)\psi \\ &= (\overline{g}\psi) \lhd g\lambda \,,\end{aligned}$$

whereas

$$\begin{aligned}a\alpha_e\psi_{[e]} \lhd (e,g) &= \overline{a}\psi \lhd (e,g) \\ &= \overline{a}\psi \lhd (e, gg^{-1}) \lhd (gg^{-1}, g) \\ &= \overline{a}\psi \lhd g\lambda \,.\end{aligned}$$

Now the injection ρ in (4.4.2) carries $(\alpha_e\psi_{[e]})$ to ψ and so $\sigma\rho$ is the identity. Also a family ϕ_e constituting an G–map $\mathcal{A} \to \mathcal{B}^{\leqslant}$ is carried by ρ to the induced map $\psi : \mathfrak{L} \to \mathcal{B}$, where $\phi_e = \alpha_e\psi_{[e]}$. But σ carries ψ precisely to this composition, and so $\rho\sigma$ is also the identity, hence in the β–transitive case, (4.4.2) and (4.4.3) exhibit a natural bijection and are inverses. Therefore $\varinjlim^{E(G)}$ is left adjoint to the *inflation* functor. The inflation functor is right adjoint to the colimit functor, $\varinjlim^{E(G)}$ hence $\varinjlim^{E(G)}$ is right exact and preserves epimorphisms. $\square$

4.4.2 Composition of colimits

If G is β–transitive, then we have seen in Proposition 4.4.2 that, for every G–module $\mathcal{A}$, the colimit $\mathfrak{L} = \varinjlim^{E(G)} \mathcal{A}$ is a $G_{\updownarrow}$–module. We can therefore form $\varinjlim^{G_{\updownarrow}} \mathfrak{L}$, with canonical maps $\psi_{[e]} : \mathfrak{L}_{[e]} \to \varinjlim^{G_{\updownarrow}}$.

Proposition 4.4.4. *The colimit* $\varinjlim^{G_{\updownarrow}} \mathfrak{L} = \varinjlim^{G_{\updownarrow}} (\varinjlim^{E(G)} \mathcal{A})$ *is naturally isomorphic to* $\varinjlim^{\mathcal{L}(G)} \mathcal{A}$*.*

Proof. We show that $\varinjlim^{G_{\updownarrow}} \mathfrak{L}$ has the universal property required of $\varinjlim^{\mathcal{L}(G)} \mathcal{A}$. As above, we have $\alpha_e : \mathcal{A}_e \to \mathfrak{L}_{[e]}$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_e \arrow[r, "\alpha_e"] \arrow[d, "\lhd(e,g)"'] & \mathfrak{L}_{[e]} \arrow[dr, bend left, "\psi_{[e]}"] \arrow[d, "\lhd g\lambda"'] & \\
\mathcal{A}_{g^{-1}g} \arrow[r, "\alpha_{g^{-1}g}"'] & \mathfrak{L}_{[g^{-1}g]} \arrow[r, bend right, "\psi_{[g^{-1}g]}"'] & \varinjlim^{G_{\updownarrow}} \mathfrak{L}
\end{tikzcd}$$

from which we extract the commutative triangles

$$\begin{tikzcd}
\mathcal{A}_e \arrow[dr, "\alpha_e \psi_{[e]}"] \arrow[dd, "\lhd(e,g)"'] & \\
& \varinjlim^{G_{\updownarrow}} \mathfrak{L} \\
\mathcal{A}_{g^{-1}g} \arrow[ur, "\alpha_{g^{-1}g} \psi_{[g^{-1}g]}"'] &
\end{tikzcd}$$

Suppose we are given a family of maps $\mu_e : \mathcal{A}_e \to \mathcal{M}$ to some abelian group $\mathcal{M}$ making commutative triangles

$$\begin{tikzcd}
\mathcal{A}_e \arrow[dr, "\mu_e"] \arrow[dd, "\lhd(e,g)"'] & \\
& \mathcal{M} \\
\mathcal{A}_{g^{-1}g} \arrow[ur, "\mu_{g^{-1}g}"'] &
\end{tikzcd}$$

In particular, for $f \leqslant e$ we have

$$\begin{tikzcd}
\mathcal{A}_e \arrow[dr, "\mu_e"] \arrow[dd, "\alpha^e_f"'] & \\
& \mathcal{M} \\
\mathcal{A}_f \arrow[ur, "\mu_f"'] &
\end{tikzcd}$$

and hence a unique family of maps $\delta_{[e]} : \mathfrak{L}_{[e]} \to \mathcal{M}$ making the diagrams

$$\begin{tikzcd}
\mathcal{A}_e \arrow[dr, "\alpha_e"'] \arrow[drr, bend left, "\mu_e"] \arrow[dd, "\alpha^e_f"'] & & \\
& \mathfrak{L}_{[e]} \arrow[r, "\delta_{[e]}"'] & \mathcal{M} \\
\mathcal{A}_f \arrow[ur, "\alpha_f"] \arrow[urr, bend right, "\mu_f"'] & &
\end{tikzcd}$$

commute.

Now consider the action of $g\lambda$ on $\overline{a} = a\alpha_e \in \mathfrak{L}_{[e]}$. From (4.4.1)

$$\begin{aligned}
(\overline{a} \lhd g\lambda)\delta_{g^{-1}g} &= (a\alpha^e_\ell \lhd (g|\ell))\alpha_z\delta_{[z]} \\
&= (a\alpha^e_\ell \lhd (g|\ell))\mu_z \,.
\end{aligned}$$

Now in $\mathcal{L}(G)$ we have $(e, \ell)(\ell, (g|\ell)) = (e, (g|\ell)) = (e, g)(g^{-1}g, z)$ and so

$$\begin{aligned}
(a\alpha^e_\ell \lhd (g|\ell))\mu_z &= (a \lhd (e|g))\alpha^{g^{-1}g}_z \mu_z \\
&= (a \lhd (e|g))\mu_{g^{-1}g} \\
&= a\mu_e \\
&= a\alpha_e\delta_{[e]} \\
&= \overline{a}\delta_{[e]}.
\end{aligned}$$

Hence the triangles

$$\begin{tikzcd}
\mathfrak{L}_{[e]} \arrow[dd, "\lhd g\lambda"'] \arrow[dr, "\delta_{[e]}"] & \\
& \mathcal{M} \\
\mathfrak{L}_z \arrow[ur, "\delta_z"'] &
\end{tikzcd}$$

commute and induce a unique map $\delta : \varinjlim^{G_\updownarrow} \mathfrak{L} \to \mathcal{M}$ making the diagram

$$\begin{tikzcd}
\mathcal{A}_e \arrow[r, "\alpha_e"] \arrow[d, "\lhd(e,g)"'] & \mathfrak{L}_{[e]} \arrow[d, "\lhd g\lambda"'] \arrow[dr, "\psi_{[e]}"] \arrow[drr, bend left, "\delta_{[e]}"] & & \\
& & \varinjlim^{G_\updownarrow} \mathfrak{L} \arrow[r, "\delta"] & \mathcal{M} \\
\mathcal{A}_{g^{-1}g} \arrow[r, "\alpha_{g^{-1}g}"'] & \mathfrak{L}_{[g^{-1}g]} \arrow[ur, "\psi_{[g^{-1}g]}"] \arrow[urr, bend right, "\delta_{[g^{-1}g]}"'] & &
\end{tikzcd}$$

commute, since $\mathfrak{L}_z = \mathfrak{L}_{g^{-1}g}$. $\square$

4.5 Homology of the level groupoid

In this section we discuss the main goal of this chapter. The homology of an ordered groupoid G with coefficients in a G–module is defined in Chapter 3. The result below is adapted from [7].

Proposition 4.5.1. *Let $G_\updownarrow$ be the levelled groupoid obtained from the ordered groupoid G. Suppose $\mathcal{B}$ is a $G_\updownarrow$ module. Then the homology functor $H_* : \mathrm{Mod}_{G_\updownarrow} \to \boldsymbol{Ab}$ satisfies*

1. *$H_0(G_\updownarrow, \mathcal{B}) = \varinjlim^{G_\updownarrow} \mathcal{B}$,*
2. *$H_n(G_\updownarrow, \mathcal{B}) = 0$ for all $n > 0$ and all projective $\mathcal{B}$.*

We proceed with the main theorem which identifies the homology groups of the ordered groupoid G with that of it associated level groupoid whenever G is β–transitive.

Theorem 4.5.2. *For any β–transitive level groupoid G and G–module $\mathcal{A}$, and any $n \geqslant 0$, the homology groups $H_n(G, \mathcal{A})$ and $H_n(G_\updownarrow, \varinjlim^{E(G)} \mathcal{A})$ are isomorphic.*

Proof. We consider the functor $\mathrm{Mod}_{\mathcal{L}(G)} \to \mathbf{Ab}$ given by

$$\mathcal{A} \to H_n(G_{\updownarrow}, \varinjlim^{E(G)} \mathcal{A})\ .$$

For $n = 0$ we have

$$H_0(G_{\updownarrow}, \varinjlim^{E(G)} \mathcal{A}) = \varinjlim^{G_{\updownarrow}} (\varinjlim^{E(G)} \mathcal{A}) \cong \varinjlim^{\mathcal{L}(G)} \mathcal{A} = H_0(G, \mathcal{A})$$

by Proposition 4.4.4. Now suppose $\mathcal{P}$ is a projective G–module. Theorem 4.4.3 shows that $\varinjlim^{E(G)}$ is left adjoint to the inflation functor. By Lemma 4.4.1 the inflation functor $\mathrm{Mod}_{G_{\updownarrow}} \to \mathrm{Mod}_{\mathcal{L}(G)}$ preserves epimorphisms (projectives), and so its left adjoint $\varinjlim^{E(G)}$ preserves projectives. Therefore $\varinjlim^{E(G)} \mathcal{P}$ is projective, and for $n > 0$ we have $H_n(G_{\updownarrow}, \varinjlim^{E(G)} \mathcal{P}) = 0$. Thus we have that $H_n(G_{\updownarrow}, \varinjlim^{E(G)} \mathcal{P})$ agrees on H_0 with $H_n(G, A)$ as well as H_n for $n > 0$ since it vanishes on projectives. Therefore $H_n(G_{\updownarrow}, \varinjlim^{E(G)} \mathcal{A})$ is isomorphic to $H_n(G, \mathcal{A})$ for any $n \geqslant 0$ as desired. $\square$

Chapter 5

Second Cohomology and Extensions of Ordered Groupoids with Abelian kernels

The principal theme of this chapter is to give an account of the connection between the second cohomology group of an ordered groupoid with an appropriate coefficient system and the set of equivalence classes of *extensions* of ordered groupoids with abelian kernel. We consider some kinds of *crossed complexes* as extensions of ordered groupoids with abelian kernel. We shall build on the classifications of ordered crossed complexes to discuss the higher cohomology theory that fits n-fold extensins for $n > 2$ in Chapter 6. The relationship between extensions and the second cohomology groups is a widely known result in applications of group cohomology theory and has appeared in several works including [22, VI section 10] and [3, IV section 3]. A closely related result to the content of this chapter is found in [25, Theorem 7.4]. The ideas of Lausch presented in [25] are the main inspiration for our study in this chapter. The correspondence has been studied for ordered groupoids by Matthews in [33] using the factor set approach taking inspiration from arguments in [25]. Our treatment in this chapter is much simpler and a conceptual approach which does not involve computing *factor sets* discussed in [33]. We shall follow the work of Gruenberg in [19, Chapter 5] to discuss the analogue of the result on inverse semigroups in [25, Theorem 7.4] in ordered groupoids. In group theory, the principal tool used in constructing the connection is the five-term exact sequence of low-dimensional cohomology groups. We have performed the parallel construction for ordered groupoids. This exact sequence of cohomology groups will then be used as the pivot for the discussion of the connection between the set of extensions of ordered groupoids with abelian kernel and the second cohomology group of ordered groupoids.

Our account of the connection is built up in the three sections of this chapter. The first deals with the application of the cohomological functor on ordered groupoids. This application yields the five-term exact sequence of cohomology groups of ordered groupoids. We will spend the second section discussing extensions of ordered groupoids with abelian kernels; the type of crossed complexes identified with the second cohomology groups discussed in section three.

5.1 Exact sequences of cohomology groups of ordered groupoids

This section discusses the preliminary story of the connection between the second cohomology groups of ordered groupoids and the set of extensions of ordered groupoids with abelian kernel. In group theory the ingredient for the discussion of the five-term exact sequence is the exact sequence of groups $N \hookrightarrow G \twoheadrightarrow Q$ where N is a normal subgroup of G and Q is the factor group G/N. This sequence is then used to generate some sequence of Q–modules which is used to deduce the five-term exact sequence (see [10]). The injection $N \hookrightarrow G$ together with the property that G acts on N by automorphisms is an example of a general concept called a *crossed module* over a group. In our discussions on the corresponding five-term exact sequence for ordered groupoids, we will make our definitions in general concepts and restrict to specific cases where necessary. This section is made of three subsections which builds up discussion on the five-term exact sequence of cohomology groups of low dimensions of ordered groupoids. We begin with a review of ordered crossed complexes and ordered chain complexes over ordered groupoids with reference to [1] inspired by [5]. The penultimate subsection presents the study of short exact sequence of modules of ordered groupoids derived from exact crossed complexes; a vital tool for the constructions of the five-term exact sequence in the last subsection.

5.1.1 Ordered crossed complexes and ordered chain complexes

We commence our discussion with the concept of crossed modules over ordered groupoids which is a generalisation of crossed modules over groups.

Definition A *crossed module* over an ordered groupoid G is a quadruple $(G, N, \mu, \triangleleft)$ defined by the following data

- N is a collection of groups N_e for $e \in G_0$,
- $\triangleleft$ is an action of G on N,
- μ is an equivariant ordered functor $N \to G$ that is $(n \triangleleft g)\mu = (n\mu) \triangleleft g$ and μ is the identity on objects,
- $n \triangleleft m\mu = m^{-1}nm$ for $m, n \in N_e$.

The latter condition is called the *Peiffer* relation. We say that the quadruple is a *precossed* module if the first three data are satisfied. The concept of crossed modules over ordered groupoids spans many situations.

Example 5.1.1 Let G be an ordered groupoid. A frequently used example of a crossed module over G is defined by the following data: the inclusion of a normal ordered subgroupoid, $N \hookrightarrow G$ and the the conjugation action of G on N. The inclusion is clearly an equivariant map.

In the case where $G_0 = \{e\}$ we obtain a crossed module over a group.

Example 5.1.2 Let $\mathcal{N}$ be a G–module. We regard $\mathcal{N}$ as the collection of abelian groups $\{\mathcal{N}_e\}_{e \in G}$ on which G acts via automorphism. Define the map $\Theta : \mathcal{N} \to G$

by $\mathcal{N}_e \mapsto e$. Then Θ is clearly an equivariant map and hence we have the crossed module $(\mathcal{N}, G, \Theta, \triangleleft)$. An *ordered morphism of crossed modules* over ordered groupoids is the pair of ordered functors $(f, \tau) : (G, N, \mu, \triangleleft) \to (G', N', \mu', \triangleleft')$ with $f : N \to N'$ and $\tau : G \to G'$ satisfying

- $f\mu' = \mu\tau$,
- $(n \triangleleft g)f = (nf) \triangleleft g\tau$.

The morphism (f, τ) is an immersion (fibration) if both f and τ are immersions (fibration). Crossed modules together with these morphisms constitute the category of crossed modules $\mathcal{CM}$ over ordered groupoids. A crossed module is a key component of the algebraic structure of a *crossed* complex. The complex is in fact a natural generalisation of crossed modules and for unordered groupoids is presented by Brown and Higgins in [5]. The applications of crossed modules play a pivotal role in discussing the properties of crossed complexes.

A crossed complex is pictorially a chain complex in higher dimensions carrying some module structure to be described latter and with the lower dimension $(n \leqslant 2)$ non-abelian. The interest of the paragraph to follow is to discuss ordered crossed complexes. We proceed with a formal definition of crossed complexes over ordered groupoids.

Definition A *crossed complex* C over an ordered groupoid G with set of objects G_0 is a sequence of ordered groupoids

$$\cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} G \rightrightarrows G_0$$

where C_n for $n \geqslant 3$ are G-modules so that $\delta_n($ for $n \geqslant 3)$ are G–equivariant maps and $C_2 \to G$ is a crossed module over G satisfying

- $C_1 = G$,
- $\delta_n \delta_{n-1} : C_n \to C_{n-2}$ for $n \geqslant 3$ is the zero map,
- the image of C_2 in G acts trivially on the G-modules, C_n, $n \geqslant 3$.

It is clear that the higher dimensional part ($n \geqslant 3$) is a chain complex. The image of the map $C_2 \to G$ acts by conjugation on C_2. The non-trivial action in C is effectively determined by the quotient ordered groupoid $G/\operatorname{im}(\delta_2)$ which is called the *fundamental* groupoid of C, denoted by $\pi_1(C)$. An example of crossed complexes over ordered grouopids is the complex $\mathscr{G}$ given by $N \to G \rightrightarrows G_0$ where N is the union of groups $\{N_e\}_{e \in G_0}$ and C_n for $n \geqslant 3$ are the trivial G–modules. If G_0 is a singleton set then a crossed complex over G is called a crossed complex over a group. It is to be noted that although G_0 is a singleton, it is not considered as an abelian group. Hence C is not a chain complex and an indication that ordered crossed complexes are not generalisations of ordered chain complexes as it appears to be. An ordered morphism $f : C \to C'$ of ordered crossed complexes is a family of ordered functors $\{f_n\}_{n \geqslant 2} : C_n \to C'_n$ and $f_1 : G \to G'$ inducing the identity map $G_0 \to G'_0$, such that $\{f_n\}$ is compatible with the maps $\delta : C_n \to C_{n-1},\ \delta' : C'_n \to C'_n$ and the actions of the fundamental groupoids. That is $(x)\delta_n f_{n-1} = (x) f_n \delta'_n$ and $(x \triangleleft g) f_n = x f_n \triangleleft g f_1$ for $x \in C_n$ and $g \in G$. Ordered crossed complexes together with the ordered morphisms form the category of ordered crossed complexes denoted by $\mathcal{OCRS}$.

We now present the concept of ordered chain complexes over ordered groupoids.

Definition An *ordered chain complex* A over an ordered groupoid G is a sequence

$$\cdots \xrightarrow{\partial_{n+1}} \mathcal{A}_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} \mathcal{A}_2 \xrightarrow{\partial_2} \mathcal{A}_1 \xrightarrow{\partial_1} \mathcal{A}_0$$

of G–modules and G–morphisms satisfying $\partial^2 = 0$. Suppose $\mu : G \to G'$ is an ordered functor and that A and A' are chain complexes over G and G' respectively. Then an ordered morphism of ordered chain complexes over μ is a family of maps $\{f_n\} : \mathcal{A}_n \to \mathcal{A}'_n$ such that $\partial f = f \partial$. Ordered chain complexes over ordered groupoids together with their ordered morphisms forms the category of ordered chain complexes over ordered groupoids denoted by $\mathcal{OCHN}$.

We note that an ordered crossed complex contains an ordered chain complex as its *trunk* (dimension $\geqslant 3$), and its *root* (dimension $\leqslant 2$) is a sequence of non-abelian objects. So now we set out to define a functor $\nabla : \mathcal{OCRS} \to \mathcal{OCHN}$ that preserves the information on the fundamental groupoid of the ordered crossed complex. The functor can be thought of as "quasi-abelianisation" of the nonabelian root of the ordered crossed complex. The goal of not losing information on the fundamental groupoid suggests some algebraic structures useful for defining ∇. In the case of unordered groupoids, Brown and Higgins in [5] discuss that the candidates are the augmentation ideal of the fundamental groupoid and the "derived" module of quotient map of the crossed complex. This is because we require that they must be modules over the fundamental groupoids and hence involve the quotient map. A merit of working with modules in this case is that, we have the merit of making use of their functorial structures. Thus we spend some time to discuss the construction of these candidate structures for ordered groupoids and hence define ∇.

Definition Suppose $\theta : G \to H$ is a groupoid map and that $\mathcal{B} \in \mathrm{Mod}_{\mathcal{L}(H)}$. A function $f : G \to \mathcal{B}$ such that $xf \in \mathcal{B}_{(x^{-1}x)\theta}$ for each $x \in G$ is called a *θ–derivation* if

$$(xy)f = (xf) \triangleleft y\theta + yf$$

whenever the product xy exists in G

An example is the derivation $G \to KG$ from the groupoid G into the augmentation ideal defined by $g \mapsto g - 1_{g^{-1}g}$. In the ordered case, we impose some extra conditions on the function f for sufficiency. We now make the formal definition of an ordered derivation for ordered morphisms.

Definition Suppose $\theta : G \to H$ is an ordered morphism of ordered groupoids and let $\mathcal{B}$ be an H–module. Then an order-preserving function $f : G \to \mathcal{B}$ is called a *θ–derivation* if

1. $gf \in \mathcal{B}_{(g^{-1}g)\theta}$
2. $(g_1g_2)f = (g_1f) \triangleleft (g_2\theta) + g_2f$, whenever the product g_1g_2 exist in G.

Now we will discuss some universal constructions. The goal is to construct an H–module relative to the ordered morphism θ together with a universal derivation map. The module appears in group theory in the form $KG \otimes_G \mathbb{Z}H$, with a comprehensive account in [11].

Definition Suppose $\theta : G \to H$ is an ordered morphism. Then its *derived* module is an H–module D_θ together with a θ–derivation δ which factorizes every θ–derivation $f : G \to \mathcal{M}$ where $\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}(H)}$: that is there is a unique module map $D_\theta \xrightarrow{\widehat{f}} \mathcal{M}$ such that $f = \delta\widehat{f}$.

The derived module is constructed as follows. Suppose $\theta : G \to H$. Then its derived module is a family of abelian groups $\{(D_\theta)_e\}_{e \in H_0}$ which admits an $\mathcal{L}(H)$–action together with a θ–derivation $\delta : G \to D_\theta$. Let F be an H–module presented as follows. F_e is the free abelian group on pairs $\{(g,h) \in G \times H : (g^{-1}g)\theta \geqslant hh^{-1} \text{ and } h^{-1}h = e\}$. We define an $\mathcal{L}(H)$–action on F by as follows. Let (g,h) be a basis element of F_e and let (e,s) be a morphism in $\mathcal{L}(H)$. Then

$$(g,h) \triangleleft (e,s) = (g, (ss^{-1}|h)s) \in F_{s^{-1}s}$$

and this gives a mapping $F_e \to F_{s^{-1}s}$. Now, we turn our attention to the defining the corresponding θ–derivation desired. Let $g_1, g_2 \in G$ such the pseudoproduct $g_1 * g_2 = (g_2g_2^{-1}|g_1)g_2$ exist. We discuss the necessary conditions for which an ordered function $f : G \to \mathcal{B}$ for $\mathcal{B} \in \mathrm{Mod}_{\mathcal{L}(H)}$ is a θ–derivation. It is required that $gf \in \mathcal{B}_{(g^{-1}g)\theta}$ and

$$(g_1 * g_2)f = ((g_2g_2^{-1}|g_1)g_2)f = (g_2g_2^{-1}|g_1)f \triangleleft g_2\theta + g_2f$$

However $(g_2g_2^{-1}|g_1) \leqslant g_1$ and so $(g_2g_2^{-1}|g_1)f \leqslant g_1f$ and that $(g_2g_2^{-1}|g_1)f = (g_1f)\phi$ where $\phi : \mathcal{B}_{(g_1^{-1}g_1)\theta} \to \mathcal{B}_{(\mathbf{r}(g_2g_2^{-1}|g_2))\theta}$ is the order map. This implies f satisfies the relation

$$\begin{aligned}
(g_1 * g_2)f = ((g_2g_2^{-1}|g_1)g_2)f &= (g_2g_2^{-1}|g_1)f \triangleleft g_2\theta + g_2f \\
&= (g_1f)\phi \triangleleft g_2\theta + g_2f \\
&= g_1f \triangleleft ((g_1^{-1}g_1)\theta, (g_2g_2^{-1})\theta) \triangleleft ((g_2g_2^{-1})\theta, g_2\theta) + g_2f
\end{aligned}$$

It follows that the necessary and sufficient condition for f to be a θ–derivation is

$$(g_1 * g_2)f = ((g_2g_2^{-1}|g_1)g_2)f \quad = \quad g_1f \lhd ((g_1^{-1}g_1)\theta, g_2\theta) + g_2f \; . \tag{5.1.1}$$

We observe that there is a natural ordered map $\ell : G \to F$ defined by $g \mapsto (g, (g^{-1}g)\theta)$. For ℓ to be a derivation we require that it satisfies (5.1.1). This requires that

$$\begin{aligned} ((g_2g_2^{-1}|g_1)g_2)\ell &= g_1\ell \lhd ((g_1^{-1}g_1)\theta, g_2\theta) + g_2\ell \\ &= (g_1, (g_1^{-1}g_1)\theta) \lhd ((g_1^{-1}g_1)\theta, g_2\theta) + (g_2, (g_2^{-1}g_2)\theta) \end{aligned}$$

and so

$$((g_2g_2^{-1}|g_1)g_2)\ell = (g_1, g_2\theta) + (g_2, (g_2^{-1}g_2)\theta) \tag{5.1.2}$$

The components of (5.1.2) are basis element of F and hence the equation may not always hold. So we act on both sides of (5.1.2) by $h \in H$ where $(g_2^{-1}g_2)\theta \geqslant hh^{-1}$ so that

$$[(g_2g_2^{-1}|g_1)g_2, h) = (g_1, (hh^{-1}|g_2\theta)h) + (g_2, h) \; . \tag{5.1.3}$$

Now we impose the relations (5.1.3) on F. The quotient of F by the relation is the derived module D_θ. The composition of the ordered map $\ell : G \to F$ with the quotient $F \to D_\theta$ is the θ–derivation $\delta : G \to D_\theta$ defined by $g \mapsto (g, (g^{-1}g)\theta)$.

Proposition 5.1.1. [1, Proposition 5.4.1] *Suppose $\theta : G \to H$ is an ordered functor with derived module D_θ together with the θ–derivation $\delta : G \to D_\theta$. If $\alpha : G \to \mathcal{M}$ is a θ–derivation for $\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}(H)}$, then there is a unique G–module morphism $\beta : D_\theta \to \mathcal{M}$ that makes the diagram*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\alpha"'] & D_\theta \arrow[dl, "\beta"] \\ \mathcal{M} & \end{tikzcd}$$

commute.

Proof. See [1] for proof □

Proposition 5.1.2. [1, Proposition 5.4.2] *Let G be an ordered groupoid. Then the augmentation module KG is the derived module of the identity map $G \to G$.*

Proposition 5.1.3. [1, Proposition 5.3.3] *The functor $\ltimes$: $\mathrm{Mod} \to \mathcal{OG}$ defined by $\mathcal{M} \mapsto G \ltimes \mathcal{M}$ where $\mathcal{M} \in \mathrm{Mod}_{\mathcal{L}(G)}$ is right adjoint to functor $K : \mathcal{OG} \to \mathrm{Mod}$ defined by the augmentation module. Hence K preserves colimits.*

Let $\mathcal{OG}^2$ be the ordered functor category of morphisms of ordered groupoids. Objects of the category $\mathcal{OG}^2$ are ordered functors of ordered groupoids and commutative squares are the morphisms. The construction of the derived module generates the functor $D : \mathcal{OG}^2 \to \mathrm{Mod}$ defined by

$$D(\theta : G \to H) = D_\theta \in \mathrm{Mod}_{\mathcal{L}(H)}$$

Proposition 5.1.4. [1, Proposition 5.5.1] *The functor D has a right adjoint $\mathcal{X}$: $\mathrm{Mod} \to \mathcal{OG}^2$ defined by $M \mapsto (G \ltimes M \xrightarrow{p} G)$*

Lemma 5.1.5. *Suppose C is a crossed complex with fundamental groupoid Q and let $G \xrightarrow{\theta} Q$ be the canonical quotient map. Then there is a sequence of Q–modules*

$$C_2^{ab} \xrightarrow{\partial_2} D_\theta \xrightarrow{\partial_1} KQ$$

such that

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "1"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "1"] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & G \arrow[r, "\theta"] \arrow[d, "\alpha_1"] & Q \arrow[d, "\alpha_0"] \\
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_3"] & C_2^{ab} \arrow[r, "\partial_2"] & D_\theta \arrow[r, "\partial_1"] & KQ
\end{tikzcd}$$

commutes and the lower line is an ordered chain complex over Q. The map α_0 is the Q–derivation defined by $q \mapsto q - 1_{q^{-1}q}$, α_1 is the universal θ–derivation and α_2 is the ordered abelianisation map.

Proof. We have that in the higher dimensions, $n \geqslant 3$, of the two sequences the modules are the same. Hence by definition $\partial^2 = 0$ as required. So it suffices to show that C_2^{ab} is a Q–module and determine the maps ∂_i for $i = \{1, 2\}$ such that $\partial_3\partial_2 = 0 = \partial_2\partial_1$, and such that the two right-hand squares commute.

Consider the poset Q_0 as an ordered groupoid and suppose $\mathcal{M} \in \mathrm{Mod}_{Q_0}$. Suppose $\xi : C_2 \to G_0$ is the trivial ordered functor. Then a ξ–derivation $C_2 \to \mathcal{M}$ is a morphism of ordered groupoids. Let D_ξ be the derived module of ξ. Then there are morphisms $D_\xi \to \mathcal{M}$ and $C_2^{ab} \to \mathcal{M}$ of abelian ordered groupoids so that

$$\begin{tikzcd} D_\xi \arrow[dr, bend right] & C_2 \arrow[l] \arrow[r] \arrow[d] & C_2^{ab} \arrow[dl, bend left] \\ & \mathcal{M} & \end{tikzcd}$$

commutes. Since D_ξ and C_2^{ab} have the same universal property, we have that D_ξ coincides with C_2^{ab} and so C_2^{ab} is Q_0–module. We show that C_2^{ab} is a Q–module. We explain the Q–action on C_2^{ab} as follows. Take $c \in C_2(e)$ and $\bar{c} = C_2^{ab}(e)$. Then for $q \in Q(e, f)$ define $\bar{c} \lhd q = \overline{(c \lhd g)}$ for $g \mapsto q$ and $g \in G(e, f)$. This is well-defined because $\mathrm{im}(\delta_2)$ acts by conjugation on C_2 from the crossed module structure and so trivially on C_2^{ab}. Therefore C_2^{ab} is a Q–module and the abelianisation map $\alpha_2 : C_2 \to C_2^{ab}$ is the universal ξ–derivation map.

We now present the map $\partial_2 : C_2^{ab} \to D_\varphi$. It is a map of derived modules and so it is the image of some commutative square in the category $\mathcal{OG}^2$ under the operator D. We have that ∂_2 is the unique map corresponding to the commutative square

$$\begin{tikzcd} C_2 \arrow[r, "\xi"] \arrow[d, "\delta_2"] & Q_0 \arrow[d] & & D_\xi = C_2^{ab} \arrow[d, "\partial_2"] \\ G \arrow[r, "\theta"] & Q \arrow[r, phantom, "\xrightarrow{D}"] & & D_\theta \end{tikzcd}$$

We show that ∂_2 is a Q-equivariant map. Let $x \in C_2^{ab}$, $q \in Q$ and that $x \lhd q$ is defined. Suppose $x = c_2\alpha_2$ and $q = g\theta$. Then we see that

$$\begin{aligned} (x \lhd q)\partial_2 &= (c_2 \lhd g)\delta_2\alpha_1 \ , \\ &= (g^{-1}c_2\delta_2 g)\alpha_1 \ , \\ &= [(g^{-1}\alpha_1) \lhd c_2\delta_2\theta + (c_2\delta_2)\alpha_1] \lhd g\theta + g\alpha_1, \qquad \text{since } \alpha_1 \text{ is a derivation,} \end{aligned}$$

but $c_2\delta_2\theta = \mathrm{id}$. Hence

$$\begin{aligned}(x \triangleleft q)\partial_2 &= (g^{-1}\alpha_1) \triangleleft g\theta + (c_2\delta_2\alpha_1) \triangleleft g\theta + g\alpha_1 \ , \\ &= -g\alpha_1 + (c_2\delta_2\alpha_1) \triangleleft g\theta + g\alpha_1 \qquad \text{since } D_\theta \text{ is abelian therefore} \\ &= (c_2\alpha_2\partial_2) \triangleleft g\theta \\ &= (x\partial_2) \triangleleft q\end{aligned}$$

as required. Now KQ is a Q–module and the composition $G \xrightarrow{\theta} Q \to KQ$ is a θ–derivation. Hence by the universal property of D_θ we obtain the Q–derivation $\partial_1 : D_\theta \to KQ$. So the diagram in the lemma commutes as desired. Finally we have that $\alpha_2\partial_2\partial_1 = \delta_2\theta\alpha_0 = 0$ and hence $\partial_2\partial_1 = 0$ since α_2 is surjective. Again $\partial_3\partial_2 = \delta_3\delta_2\alpha_1 = 0$ since the image of δ_2 is the kernel of θ and α_1 is a θ–derivation. Hence the lower chain is an ordered chain complex over Q as desired. □

Now we make a formal definition of the functor $\nabla : \mathcal{OCRS} \to \mathcal{OCHN}$.

Definition Let C be an ordered crossed complex with fundamental groupoid denoted by Q. Then $\nabla(C)$ is the ordered chain complex

$$\cdots \to C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2^{ab} \xrightarrow{\partial_2} D_\varphi \xrightarrow{\partial_1} KQ$$

of modules over Q.

Proposition 5.1.6. [1, Proposition 5.7.4] *The functor $\nabla : \mathcal{OCRS} \to \mathcal{OCHNS}$ is left adjoint to the functor $\Theta : \mathcal{OCHN} \to \mathcal{OCRS}$ defined by $A \mapsto \Theta(A)$ where $\Theta(A)$ is defined by*

$$\cdots \to \mathcal{A}_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_4} \mathcal{A}_3 \xrightarrow{\delta_3} \mathcal{A}_2 \xrightarrow{(0,\delta_2)} G \ltimes \mathcal{A}_1 \ .$$

5.1.2 Exact sequences

This section is devoted to studying the lifting properties of the functor $\nabla : \mathcal{OCRS} \to \mathcal{OCHN}$. The principal result is contained in Lemma 5.1.7. In [11] and [10], Crowell constructs a sequence of modules from a sequence of groups

$$K \to G \to H \tag{5.1.4}$$

and shows that the exactness of the sequence (5.1.4) is lifted by his construction to the corresponding sequence of H–modules. Brown and Higgins in [5] extend Crowells' construction of the module sequence from the sequence of groups to unordered groupoids. They show that that given a crossed complex C over a groupoid G, the functor that associates C the corresponding sequence of modules ∇C preserves exactness.

In the preceding section we presented the analogous construction of a sequence of modules from a sequence of ordered groupoids. Now we shall present an extension of the exact module sequence presented in [5] by Brown and Higgins.

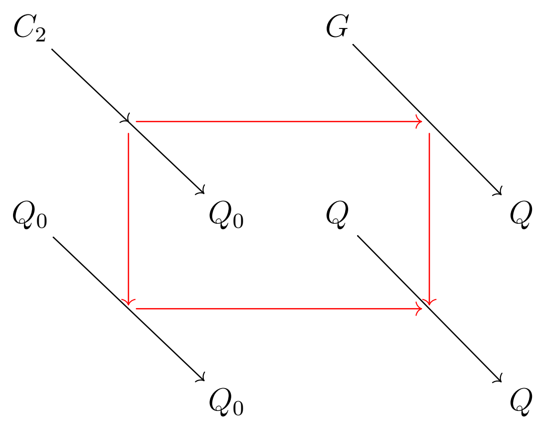
Let C be a crossed complex over the ordered groupoid G. Denote the fundamental groupoid of C by Q. By Lemma 5.1.5 there exist an ordered chain complex of modules over Q that is associated with the sequence $C \to Q$ by ∇. We show in the following lemma that ∇ preserves exactness of the sequence $C \to Q$.

Lemma 5.1.7. *Suppose that C is an ordered crossed complex and let $G \xrightarrow{\varphi} Q$ be the quotient map. If the sequence of ordered groupoids $C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} G \xrightarrow{\varphi} Q$ is exact then the corresponding sequence $C_3 \xrightarrow{\partial_3} C_2^{ab} \xrightarrow{\partial_2} D_\varphi \xrightarrow{\partial_1} KQ$ of Q–modules is exact.*

Proof. By exactness we mean $\ker(\varphi) = \operatorname{im}(\delta_2)$ and $\ker(\delta_2) = \operatorname{im}(\delta_3)$. Thus the fact that the short sequence $C_2 \xrightarrow{\delta_2} G \xrightarrow{\varphi} Q$ is exact implies that $\ker(\varphi) = \operatorname{im}(\delta_2)$. We note that $Q \cong G/\operatorname{im}(\delta_2)$. Thus the cokernel of δ_2 is defined as the pushout,

$$\begin{tikzcd}
C_2 \arrow[r, "\delta_2"] \arrow[d] & G \arrow[d, "\varphi"] \\
Q_0 \arrow[r] & Q
\end{tikzcd}$$

and consequently the sequence



is a pushout square in $\mathcal{OG}^2$. From Proposition 5.1.4 the functor D is left adjoint to the functor $\mathcal{X}$ and hence D preserves colimits. And so if we apply D to the pushout square we obtain the diagram

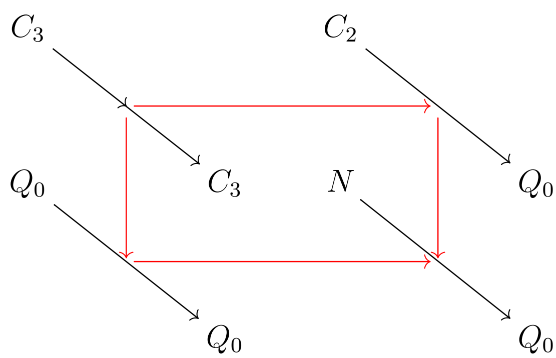
$$\begin{tikzcd}
C_2^{ab} \arrow[r] \arrow[d] & D_\varphi \arrow[d] \\
0 \arrow[r] & KQ
\end{tikzcd}$$

which is a pushout square in the category Mod_Q. Therefore the sequence

$$C_2^{ab} \to D_\varphi \to KQ$$

is exact.

Now we need to show that the sequence of Q–modules $C_3 \to C_2^{ab} \xrightarrow{\partial_2} D_\varphi$ is exact. Let N be the kernel of the map φ. Then the exactness of $C_3 \to C_2 \to G \xrightarrow{\varphi} Q$ implies that the sequence $C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\tilde{\partial}_2} N$ is exact. That is $\mathrm{im}(\partial_3) = \ker(\tilde{\partial}_2)$. This implies a pushout square



in $\mathcal{OG}^2$ and hence

$$\begin{array}{ccc} C_3 & \longrightarrow & C_2^{ab} \\ \big\downarrow & & \big\downarrow \tilde{\partial} \\ 0 & \longrightarrow & N^{ab} \end{array}$$

is a pushout square in Mod_Q. Therefore the sequence $C_3 \xrightarrow{\partial_3} C_2^{ab} \to N^{ab}$ is exact. It thus suffices to show that map $N^{ab} \to D_\varphi$ is injective.

Suppose $e \in Q_0$. Then the ordered derived module over the ordered functor $G \xrightarrow{\varphi} Q$ is defined by $(D_\varphi)_e$ is a quotient of the free abelian group on pairs (g, q) where $g \in G$ and $q \in Q$ such that $(g^{-1}g)\varphi \geqslant qq^{-1}$ and $q^{-1}q = e$ and is subject to the relation

$$\langle h * g, q\rangle = \langle h, g\varphi * q\rangle + \langle g, q\rangle \tag{5.1.5}$$

whenever the product $h * g$ exist in G. D_φ is an Q–module hence admits an $L(Q)$–action as part of the construction. We define the action of the morphism (e, q') on a basis element $\langle g, q\rangle$ is defined as

$$\langle g, q\rangle \triangleleft (e, q') = \langle g, q * q'\rangle \ .$$

It is clear that the trivial groupoid Q_0 acts trivially on D_φ. Now suppose $k\varphi = q$ for some $k \in G$. Then the product $g\varphi * k\varphi$ is defined. Since φ is identity on objects, the product $g * k$ is defined. Using equation (5.1.5), the generators of $(D_\varphi)_e$ can be written as

$$\langle g, q\rangle = \langle g, k\varphi\rangle = \langle g * k, e\rangle - \langle k, e\rangle$$

since $k\varphi * e = k\varphi$. In particular we write $[l] = \langle l, (l^{-1}l)\varphi\rangle$ for any $l \in G(-, e)$. This notation decomposes the generator $\langle g, q\rangle$ into the sum of generators

$$\langle g, q\rangle = [g * k] - [k] \ .$$

The relation in equation (5.1.5) now reads

$$[h * g * k_1] - [k_1] = [h * g * k_2] - [g * k_2] + [g * k_3] - [k_3] \tag{5.1.6}$$

where we may chosen distinct preimages k_i for q, since φ is the identity on objects, the k_i are coterminal. Hence we deduce from equation (5.1.5) the relation

$$[h * g] \;=\; [h] + [g] \quad \text{for } g \in N_{h^{-1}h}. \tag{5.1.7}$$

We note that for $a\varphi = b\varphi$ we have $a^{-1}b \in N$ and hence $[b] = [a \cdot a^{-1}b] = [a] + [a^{-1}b]$ from equation (5.1.7). From equation (5.1.6) we have

$$[h * g * k_1] - [h * g * k_2] = [k_1] - [g * k_2] + [g * k_3] - [k_3]$$

and so using the relation in equation (5.1.7) and fact that $[a]-[b] = -[a^{-1}b]$ whenever $a\varphi = b\varphi$ we get that

$$\begin{aligned}
[h * g * k_1] - [h * g * k_2] &= [h * g] + [k_1] - [h * g] - [k_2] = -[k_1^{-1}k_2] \\
[g * k_3] - [g * k_2] &= [g] + [k_3] - [g] - [k_2] = -[k_3^{-1}k_2] \\
[k_1] - [k_3] &= -[k_1^{-1}k_3]
\end{aligned}$$

Therefore

$$\begin{aligned}
[h * g * k_1] - [h * g * k_2] &= [k_1] - [g * k_2] + [g * k_3] - [k_3] \\
[k_1^{-1}k_2] &= [k_3^{-1}k_2] + [k_1^{-1}k_3] \\
-[k_1] + [k_2] &= -[k_3] + [k_2] - [k_1] + [k_3] \\
[k_2] &= [k_2]
\end{aligned}$$

which implies equation (5.1.6) holds. So equation (5.1.7) is a sufficient defining relation on the generators $[l]$ of D_φ. Hence it is natural to define the ordered derived module over φ by the presentation

$$(D_\varphi)_e := \big[[g] : (g^{-1}g)\varphi \geqslant e \mid [g * n] = [g] + [n] \text{ for } g \in G(-, e) \text{ and } n \in N_e\big] \;.$$

Any $g \in G$ can be written $g = t(g)s(g)$ for some coset representative $t(g) \in G$ and $s(g) \in N$. We select $\mathbf{1}_e$ to represent $n \in N_e$ and hence write $n = \mathbf{1}_e \cdot n$ for $n \in N_e$.

We obtain the map $\bar{s} : D_\varphi \to N^{ab}$; $[g] \mapsto (s(g))\alpha_2$ where α_2 is the abelianisation map. The map $\bar{s}$ is an abelian group homomorphism as it preserves the relation in D_φ which we now show. Consider $g * n$ where $n \in N_{g^{-1}g}$. Then we have that $g * n = t(g * n)s(g * n) = t(g)t(n)s(g)(n) = t(g)\mathbf{1}_e s(g)s(n) = t(g)s(g)n$ since $n \in N$ and so $s(n) = n$. Therefore $t(g) = t(g * n)$ and $s(g)n = s(g * n)$. This leads to

$$\bar{s}([g * n]) = (s(g * n))\alpha_2 = (s(g) \cdot n)\alpha_2 = (s(g))\alpha_2 + (n)\alpha_2 = \bar{s}([g]) + \bar{s}([n])$$

as desired. The homomorphism $\bar{s}$ is in fact the left inverse of $N^{ab} \xrightarrow{\gamma} D_\varphi$ induced by $C_2^{ab} \xrightarrow{\partial_2} D_\varphi$, that is the composition $N^{ab} \xrightarrow{\gamma} D_\varphi \xrightarrow{\bar{s}} N^{ab}$ is the identity. Let $n\alpha_2 = v \in N^{ab}$. Then

$$(v\gamma)\bar{s} = ((n\alpha_2)\gamma)\bar{s} = (n\alpha_1)\bar{s} = ((n\alpha_1)s)\alpha_2 = n\alpha_2 = v\ .$$

Therefore γ is injective. We have the following sequence of modules

$$\begin{tikzcd}
C_3 \arrow[r, "\partial_3"] \arrow[rrd, bend right, red] & C_2^{ab} \arrow[r, "\partial_2"] \arrow[rd, "\tilde{\partial}_2"] \arrow[rr, bend left, red] & D_\varphi \arrow[r, "\partial_1"] & KQ \\
& & N^{ab} \arrow[u, "\gamma"] &
\end{tikzcd}$$

where the red arrows indicate the directions of exactness shown so far.

We show now show that $\ker(\partial_1) = \mathrm{im}(\gamma)$. Suppose $\sum[g_i] \in \ker(\partial_1)$ so

$$\sum[g_i] \mapsto \sum(g\varphi - e) \in (KQ)_e$$

such that $\sum(g\varphi - e) = 0$. Thus we have either $g_i\varphi = e$ which implies $g_i = n_i \in N_e$ or some $g\varphi$ and $h\varphi$ cancel out in the summation so $h^{-1}g \in N_e$.

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "h"] \arrow[r, bend right, "g"'] & \bullet^{e}
\end{tikzcd}$$

We have that $g = h(h^{-1}g)$ and so $[g] = [h] + [h^{-1}g] \in (D_\varphi)_e$ and $[g] - [h] = [h^{-1}g] \in \mathrm{im}(\gamma)$. Therefore $\ker(\partial_1) = \mathrm{im}(\gamma)$. Since γ factorizes ∂_2 it follows that the sequence

$C_3 \xrightarrow{\partial_3} C_2^{ab} \xrightarrow{\partial_2} D_\varphi \xrightarrow{\partial_1} KQ$ is exact as desired.

□

Corollary 5.1.8. *Suppose $N \xrightarrow{\mu} G$ is an ordered crossed module, and let $Q = G/\operatorname{im}\mu$ with the canonical quotient map $G \xrightarrow{\varphi} Q$. Then the sequence of Q–modules $N^{ab} \to D_\varphi \to K(Q)$ is exact.*

5.1.3 Five-term exact sequence

In [11], Crowell presents an application of the exact sequence of modules from the exact sequence of groups. This subsection is aimed at presenting an application of the cohomological machinery on the exact sequence of modules of ordered groupoids constructed in the preceding subsection. We state the result in the following theorem.

Theorem 5.1.9. *Let $N \xrightarrow{\mu} G$ be an ordered crossed module, and let $Q = G/\operatorname{im}(\mu)$ so that the canonical quotient map $G \xrightarrow{\varphi} Q$ together with μ is an exact sequence of ordered groupoids. Suppose $\mathcal{A}$ is a Q-module. Then the five-term sequence*

$$0 \to \operatorname{Der}(Q, \mathcal{A}) \to \operatorname{Der}_\varphi(G, \mathcal{A}) \to \operatorname{Hom}_Q(N^{ab}, \mathcal{A}) \to H^2(Q^I, \mathcal{A}^0) \to H^2(G^I, \mathcal{A}^0)$$

is exact

Proof. The sequence $N \to G \xrightarrow{\varphi} Q$ is an exact sequence of ordered groupoids and a consequence of Lemma 5.1.7 is that the sequence of Q-modules $N^{ab} \to D_\varphi \to KQ$ is in fact exact. Applying the contravariant functor, $\operatorname{Ext}_Q(-, \mathcal{A})$, generates the exact Hom–Ext sequence

$$0 \to \operatorname{Hom}_Q(KQ, \mathcal{A}) \to \operatorname{Hom}_Q(D_\varphi, \mathcal{A}) \to \operatorname{Hom}_Q(N^{ab}, \mathcal{A}) \to \operatorname{Ext}^1_Q(KQ, \mathcal{A}) \to \operatorname{Ext}^1_Q(D_\varphi, \mathcal{A})$$

in low dimensions. We make the following identifications.

- The module KQ is the derived module relative to the identity map on Q. The surjection $\psi : Q \to KQ$ defined by $q \mapsto q - 1_{q^{-1}q}$ is the universal derivation and hence composition with homomorphisms from KQ to $\mathcal{A}$ is essentially a derivation, therefore the isomorphism $\mathrm{Hom}_Q(KQ, \mathcal{A}) \cong \mathrm{Der}(Q, \mathcal{A})$.

- Also D_φ is the φ–derived module together with the derivation $G \to D_\varphi$. Homomorphisms from D_φ to $\mathcal{A}$ compose with $G \to D_\varphi$ to give φ–derivations. Hence the isomorphism $\mathrm{Hom}_Q(D_\varphi, \mathcal{A}) \cong \mathrm{Der}_\varphi(G, \mathcal{A})$.

- From Theorem 3.4.4 we get the isomorphism $\mathrm{Ext}^1_Q(KQ, \mathcal{A}) \cong H^2(Q^I, \mathcal{A}^0)$.

- Finally $G \to D_\varphi$ is the universal φ–derivation and so factorizes every φ–derivation $G \to \mathcal{A}$. However the module $\mathcal{A}$ is a G–module with the G–action defined via φ. Hence we obtain the φ–derivation $KG \to \mathcal{A}$. Since $G \to D_\varphi$ is the universal φ–derivation we obtain a unique map $KG \xrightarrow{\beta} D_\varphi$ which induces a morphism $\mathrm{Ext}^i_Q(D_\varphi, \mathcal{A}) \to \mathrm{Ext}^i_G(KG, \mathcal{A})$. We show that the induced map is injective for $i = 1$.

  We proceed by embedding $\mathcal{A}$ into some injective Q-module $\mathcal{I}$ and set $\mathcal{C}$ to be the quotient module. Applying the covariant functors $\mathrm{Ext}_G(KG, -)$ and $\mathrm{Ext}_Q(D_\varphi, -)$ to the exact sequence $\mathcal{A} \to \mathcal{I} \to \mathcal{C}$ gives the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^0_G(KG,\mathcal{A}) \arrow[r] & \mathrm{Ext}^0_G(KG,\mathcal{I}) \arrow[r] & \mathrm{Ext}^0_G(KG,\mathcal{C}) \arrow[r] & \mathrm{Ext}^1_G(KG,\mathcal{A}) \arrow[r] & \mathrm{Ext}^1_G(KG,\mathcal{I}) \\
\mathrm{Ext}^0_Q(D_\varphi,\mathcal{A}) \arrow[r] \arrow[u] & \mathrm{Ext}^0_Q(D_\varphi,\mathcal{I}) \arrow[r] \arrow[u] & \mathrm{Ext}^0_Q(D_\varphi,\mathcal{C}) \arrow[r] \arrow[u] & \mathrm{Ext}^1_Q(D_\varphi,\mathcal{A}) \arrow[r] \arrow[u, "\tau"] & \mathrm{Ext}^1_Q(D_\varphi,\mathcal{I}) \arrow[u, "0"]
\end{tikzcd}$$

  where the vertical maps are the induced map. Note that $\mathrm{Ext}^0_Q(D_\varphi, -) = \mathrm{Hom}_Q(D_\varphi, -)$ and $\mathrm{Ext}^0_G(KG, -) = \mathrm{Hom}_G(KG, -)$. Thus a Q–morphism $D_\varphi \to (-)$ corresponds to a φ–derivation $G \to (-)$. By the universal properties of the derived module KG we get the corresponding derivation $KG \to (-)$. Also every φ–derivation $KG \to (-)$ determines a φ–derivation $G \to (-)$ which is determined by the Q–map $D_\varphi \to (-)$. Thus the first three vertical maps $\mathrm{Ext}^0_Q(D_\varphi, -) \to \mathrm{Ext}^0_Q(KG, -)$ are equalities.

  Now by definition $\mathrm{Ext}^1_Q(D_\varphi, \mathcal{I}) = 0$ since $\mathcal{I}$ is an injective Q–module and so $\mathrm{Ext}^1_Q(D_\varphi, \mathcal{I}) \to \mathrm{Ext}^1_G(KG, \mathcal{I})$ is a zero map.

We show that $\mathrm{Ext}^1_Q(D_\varphi, \mathcal{A}) \xrightarrow{\tau} \mathrm{Ext}^1_G(KG, \mathcal{A})$ is injective. Let $z \in \mathrm{Ext}^1_Q(D_\varphi, \mathcal{A})$ such $z \in \ker(\tau)$. Then we have that $z\tau \in \mathrm{Ext}^1_G(KG, \mathcal{A})$ is the image of some $\tilde{z} \in \mathrm{Ext}^0_G(KG, \mathcal{C})$. By exactness of the upper line we have that $\tilde{z}$ is the image of some $\widehat{z} \in \mathrm{Ext}^0_G(KG, \mathcal{I})$. By the equalities of the vertical maps discussed earlier we get that $z = 0$. Therefore $\mathrm{Ext}^1_Q(D_\varphi, \mathcal{A}) \xrightarrow{\tau} \mathrm{Ext}^1_G(KG, \mathcal{A})$ is injective as desired. Therefore making the identification $H^2(G^I, \mathcal{A}^0) \cong \mathrm{Ext}^1_G(KG, \mathcal{A})$ we get $\mathrm{Ext}^1_Q(D_\varphi, \mathcal{A}) \hookrightarrow H^2(G^I, \mathcal{A}^0)$.

Using the listed identifications we obtain the exact sequence

$$0 \to \mathrm{Der}(Q, \mathcal{A}) \to \mathrm{Der}_\varphi(G, \mathcal{A}) \to \mathrm{Hom}_Q(N^{ab}, \mathcal{A}) \to H^2(Q^I, \mathcal{A}^0) \to H^2(G^I, \mathcal{A}^0)\ .$$

□

5.2 Extensions of ordered groupoids

The theory of *extensions* of algebraic structures such as groups, regular semigroups, and groupoids has been a well studied using the (co)homological machinery on these objects. Authors such as Lausch and Loganathan have made remarkable contributions in this subject on inverse semigroups. We follow [5] to discuss the concept of extensions of ordered groupoids with abelian kernel.

Definition Let $\varphi : G \xrightarrow{\varphi} Q$ be an ordered morphism and suppose that the totally disconnected ordered groupoid N (that is N is a union of groups) is isomorphic to the kernel of φ. Then $N \to G$ a is crossed module. The exact sequence of ordered groupoids

$$N \to G \xrightarrow{\varphi} Q$$

is called an *extension* of N by Q. A morphism of extensions of N by Q is an ordered morphism of ordered groupoids $\mu : G \to G'$ such that the diagram

$$\begin{tikzcd} & G \arrow[dr] \arrow[dd, "\mu"] & \\ N \arrow[ur, hook] \arrow[dr, hook] & & Q \\ & G' \arrow[ur] & \end{tikzcd}$$

commutes. Morphisms of extensions generates a equivalence relation on the set of extensions of N by Q. We say that a pair of extensions are equivalent if we can find a morphism of extensions between them. The set of equivalent classes of extensions of N by Q is nonempty. A trivial but vital class which always exist is the class containing $N \to Q \ltimes N \to Q$ (see [1, section 4.2]). This extension is often called the *split* extension in other literatures.

5.3 Correspondence of extensions of ordered groupoids with abelian kernel and second cohomology group

This final section is devoted to showing the relationship between the set of equivalence classes of extensions of ordered groupoids with abelian kernel and the second cohomology group of ordered groupoids with coefficients in a module over the ordered groupoid. The existence of the relationship was suggested by the results in [13] which was interpreted by Lausch in [25] for inverse semigroups. The principal source of our approach to showing the connexion is [19]. We make a necessary categorization in Proposition 5.3.9 under the heading classification of extensions by $\mathbb{F}$–maps and prove the main result in Theorem 5.3.11 under the last caption of this section. We review some ideas in [42] which are essential to our later discussions. We begin with that of *free ordered groupoids*. The reference [21] gives a comprehensible account for the unordered case.

A *graph* Γ comprise of the following data: a set $\mathrm{Ob}\,(\Gamma)$ of objects, a set $E(\Gamma)$ of arrows and three maps

- an involution $E(\Gamma) \to E(\Gamma); g \mapsto g^{-1}$,
- source map $\mathbf{d} : E(\Gamma) \to \mathrm{Ob}\,(\Gamma)$ associating $g \in E(\Gamma)$ with the object at the beginning of g and,
- a target map $\mathbf{r} : E(\Gamma) \to \mathrm{Ob}\,(\Gamma)$ associating $g \in E(\Gamma)$ with the object at the end of g.

Definition An *ordered graph* Γ is a graph such that the following data are satisfied.

*A*1 $\mathrm{Ob}\,(\Gamma)$ and $E(\Gamma)$ are posets,

*A*2 the source, target and involution are order preserving maps,

*A*3 if g is an arrow and e is an object with $e \leqslant g\mathbf{d}$, there exists a unique arrow called the restriction of g to e written $(g|e)$ such that $(g|e) \leqslant g$ and $(g|e)\mathbf{d} = e$.

Suppose $e \leqslant g\mathbf{d}$, then $(g|e)^{-1}$ is accordingly defined as the unique arrow $(g^{-1}|(g|e)\mathbf{r})$. Denote by star(e) the set of all arrows with source e. Then the restriction induces a map star$(e) \to$ star(f) defined by $g \mapsto (g|f)$ whenever $e \geqslant f$ for $e, f \in E(\Gamma)$. A morphism of ordered graphs is an order preserving graph-map. Ordered graphs together with morphisms of ordered graphs constitute the category of ordered graphs $\mathscr{O}G$.

Free ordered groupoids

Let Γ be a graph. A *path* p of length n in Γ is a sequence $p = g_1, \cdots, g_n$ of arrows in Γ such that $g_i\mathbf{r} = g_{i+1}\mathbf{d}$ for $i = 1, \cdots, n-1$. We allow the empty path of length 0 at each object, denoted by $()_e$ and make use of the notations of source and target maps to describe the source and target maps on paths. Suppose p and p' are paths such that $p\mathbf{r} = p'\mathbf{d}$ then the composition pp' is defined by concatenation of paths p and p'. Let p be a path and that $e \leqslant p\mathbf{d}$ for $e \in \mathrm{Ob}\,(\Gamma)$. Then there is a unique path called the restriction of p to e denoted by $(p|e) = y_1, \cdots, y_n$ defined as follows. We set $y_1 = (g_1|e)$, $y_2 = (g_2|(g_1|e)\mathbf{r}) = (g_2|y_1\mathbf{r})$,

$y_3 = (g_3|(g_2|(g_1|e)\mathbf{r})\mathbf{r}) = (g_3|y_2\mathbf{r})$ and in general $y_i = (g_i|y_{i-1}\mathbf{r})$ for $i = 2, \cdots, n$.

Clearly $(p|e)\mathbf{r} = (g_n|y_{n-1}\mathbf{r}) \leqslant g_n\mathbf{r} = p\mathbf{r}$.

Lemma 5.3.1. [42, Lemma 5.1] *Suppose $p = g_1, \cdots, g_n$ and $p' = y_1, \cdots, y_n$ are paths in the ordered graph Γ such that p and p' are composable and $e \leqslant p\boldsymbol{d}$ for $e \in \mathrm{Ob}(\Gamma)$. Then*

1. $(pp'|e) = (p|e)(p'|(p|e)\boldsymbol{r})$,

2. *the restriction of p^{-1} to $(p|e)\boldsymbol{r}$ is given by $(p|e)^{-1}$.* □

If $g_{i+1} = g_i^{-1}$ for some i in a path p, then $p' = g_1, \cdots, g_{i-1}, g_{i+2}, \cdots, g_n$ is a path coterminal to p. We say that p and p' are *homotopic* and call the removal or inserting of the subpaths g_ig_{i+1} an *elementary homotopy*. Thus we say any two paths are homotopic if they are related by a finite sequence of elementary homotopy. This generates an equivalence relation on paths in Γ. We denote by $[p]$ the equivalence class containing the path p and define composition of equivalence classes of paths by $[p][p'] = [pp']$ whenever p and p' are composable paths.

Corollary 5.3.2. [42, Corollary 5.3] *Suppose p and p' are homotopic paths in an ordered graph Γ and $e \leqslant p\boldsymbol{d}$. Then $(p|e)$ and $(p'|e)$ are homotopic.*

Proposition 5.3.3. [42, Proposition 5.4] *Let Γ be an ordered graph. The set of homotopy equivalent classes of paths in Γ together with the composition of equivalent classes of paths is an ordered groupoid.*

Proof. The axioms $OG1$ and $OG2$ follows from Lemma 5.3.1 and we get $OG3$ and $OG4$ from Corollary 5.3.2. □

We denote this groupoid by $\pi_1(\Gamma)$ and call it the *fundamental ordered groupoid* of Γ. It is clear that there is a canonical ordered graph map $\Gamma \xrightarrow{\tau} \pi_1(\Gamma)$ defined by

$$g \mapsto [g].$$

Proposition 5.3.4. *Let Γ be an ordered graph, G an ordered groupoid and $\Gamma \xrightarrow{\theta} G$ an ordered morphism of ordered graphs. Then there is a unique ordered morphism of ordered groupoids $\pi_1(\Gamma) \xrightarrow{\theta^*} G$ extending θ.*

It follows from Proposition 5.3.4 that $\pi_1(\Gamma)$ is the *free ordered groupoid* on Γ.

Corollary 5.3.5. *$\pi_1 : \mathscr{O}G \to$ **OGpd** is left adjoint to the forgetful functor **OGpd** $\to \mathscr{O}G$.*

Let Q be an ordered groupoid and $\mathbb{F}(Q)$ the free ordered groupoid on the underlying graph of Q. The element of $\mathbb{F}(Q)$ that corresponds to an arrow $q \in Q$ will be denoted by $\lfloor q \rfloor$, and the canonical map $\pi : \mathbb{F}(Q) \to Q$ is defined by $\lfloor q \rfloor \mapsto q$. Let $\mathbb{N}(Q)$ be the kernel of π. If $w = \lfloor q_1 \rfloor \lfloor q_2 \rfloor \cdots \lfloor q_m \rfloor \in \mathbb{N}(Q)$ then $q_1 \cdots q_m \in Q_0$ and so $q_1\mathbf{d} = q_m\mathbf{r}$ and so $w\mathbf{d} = w\mathbf{r}$. Therefore $\mathbb{N}(Q)$ is a union of groups.

If $\mathcal{A}$ is a Q–module (which we shall write additively) then it is also an $\mathbb{F}(Q)$–module, with $\mathbb{F}(Q)$ acting via π. So we may form the semidirect product $S = \mathbb{F}(Q) \ltimes \mathcal{A}$. Then $T = \mathbb{N}(Q) \ltimes \mathcal{A}$ is a normal ordered subgroupoid of S, and since $\mathbb{N}(Q)$ acts trivially, T is just the pullback

$$T = \{(w, a) \in \mathbb{N}(Q) \times \mathcal{A} : w\mathbf{r} = a\mathbf{d}\}$$

with componentwise composition.

Now $\mathbb{F}(Q)$ acts by conjugation on $\mathbb{N}(Q)$ and, as above, on $\mathcal{A}$ via π. Hence we can consider the set of ordered functors $\mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ that respects the $\mathbb{F}(Q)$–actions. This is an abelian group under the operation of pointwise addition in $\mathcal{A}$, so that for $\phi, \alpha \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ we have $w(\phi + \alpha) = w\phi + w\alpha$. The notation $\mathbf{OGpd}_{(-)}(-,-)$ is used here and henceforth for $\mathrm{Hom}_{(-)}(-,-)$ for convenience to emphasise that the homomorphisms considered are ordered functors of ordered groupoids.

5.3.1 Classification of extensions by $\mathbb{F}$–maps

Given $\phi \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$, we define

$$M_\phi = \{(w, w\phi) : w \in \mathbb{N}(Q)\} \subseteq \mathbb{N} \ltimes \mathcal{A}\,.$$

Lemma 5.3.6. *M_ϕ is a normal ordered subgroupoid of S and there is an extension*

$$\mathscr{E}_\phi : \mathcal{A} \xrightarrow{\iota} S \mathbin{/\!\!/} M_\phi \xrightarrow{\pi_\phi} Q$$

of ordered groupoids.

Proof. We first show that M_ϕ is a normal ordered subgroupoid of S. That is M_ϕ satisfies the axioms $N01$–$N03$ stated in Chapter 1. Thus we have the following:

N01. The map ϕ is identity on objects and M_ϕ has same set of object as S. It is clear that M_ϕ is a subgroupoid of S and is *wide* in S.

N02. Suppose $w \in \mathbb{N}(Q)$ and $e \in S_0$ such that $e \leqslant w\mathbf{d}$. Then we define the restriction of $(w, w\phi)$ to e by $((w, w\phi)|e) = ((w|e), (w\phi|e))$. However $(w|e)\phi = (w\phi|e\phi) = (w\phi|e)$ since $((w|e)\phi)\mathbf{d} = e = e\phi$ and $(w|e)\phi \leqslant w\phi$. Therefore

$$((w, w\phi)|e) = ((w|e), (w|e)\phi) \in M_\phi$$

as desired.

N03. Let $(w, w\phi) \in M_\phi$ and $(h, a), (k, b) \in S$ such that (h, a) and (k, b) have an upper bound $(g, c) \in S$ and let $(h, a)^{-1}(w, w\phi)(k, b)$ be defined in S. Then we show that $(h, a)^{-1}(w, w\phi)(k, b) \in M_\phi$. Since $(h, a)^{-1}(w, w\phi)(k, b)$ is defined we have that

$$h\mathbf{d} = (a \lhd h^{-1})\mathbf{d} = w\mathbf{d} = (w\phi \lhd w^{-1})\mathbf{d} = k\mathbf{d} = (b \lhd k^{-1})\mathbf{d}\ .$$

But $\mathbb{N}(Q)$ acts trivially on $\mathcal{A}$ and so $w\phi \lhd w^{-1} = (www^{-1})\phi = w\phi$. The subgroupoid M_ϕ is a disjoint union of groups, hence $(h, a)^{-1}(w, w\phi)(k, b)$ defined implies $(h, a)\mathbf{d} = (k, b)\mathbf{d}$ and (h, a) and (k, b) are restrictions of (g, c) in S and so by uniqueness of restrictions, $(h, a) = (k, b)$. Thus

$$\begin{aligned}(h,a)^{-1}(w,w\phi)(k,b) &= (h,a)^{-1}(w,w\phi)(h,a)\\ &= (h^{-1}, -a \lhd h^{-1})(w,w\phi)(h,a)\\ &= (h^{-1}, -a \lhd h^{-1})(wh, (w\phi \lhd h) + a)\\ &= (h^{-1}wh, (-a \lhd h^{-1}wh) + (w\phi \lhd h) + a)\end{aligned}$$

but $-a \lhd h^{-1}wh = -a$ since $h^{-1}wh$ acts trivially and so the second component reads $-a + w\phi \lhd h + a = w\phi \lhd h = (h^{-1}wh)\phi$. Therefore

$$(h,a)^{-1}(w,w\phi)(k,b) = (h^{-1}wh, (h^{-1}wh)\phi) \in M_\phi$$

as desired.

Therefore M_ϕ is a normal ordered subgroupoid of S.

We show that the $\mathscr{E}_\phi$ is an extension of ordered groupoids. We have shown that M_ϕ is a normal ordered subgroupoid of S. This gives a quotient $S /\!\!/ M_\phi$ from section 1.1. Define the map $S /\!\!/ M_\phi \xrightarrow{\pi_\phi} Q$ by

$$[(x,a)] \mapsto x\pi \ .$$

It is evident that π_ϕ is an ordered functor of ordered groupoids. We show that π_ϕ is well defined. Let $(w, a) \in S$, then $[(w, a)]\pi_\phi = w\pi$. Suppose $(u, u\phi)(w, a)(v, v\phi)$ is defined in S. Then we have

$$\begin{aligned}(u,u\phi)(w,a)(v,v\phi) &= (u,u\phi)(wv, a \lhd v + v\phi)\\ &= (uwv, u\phi \lhd wv + (a \lhd v) + v\phi)\end{aligned}$$

and so applying π_ϕ gives

$$[(u,u\phi)(w,a)(v,v\phi)] \mapsto (uwv)\pi = w\pi = [(w,a)]\pi_\phi$$

so π_ϕ is well-defined and $[w,a]_{\pi_\phi} \in E(Q) \Leftrightarrow w \in \mathbb{N}(Q)$.

Now map $\mathcal{A} \xrightarrow{\iota} S \mathbin{/\!\!/} M_\phi$ by

$$a \mapsto [(e, a)]$$

for $a \in \mathcal{A}_e$. To show injectivity of ι let $[(f, b)] \in S \mathbin{/\!\!/} M_\phi$ with $b \in \mathcal{A}_f$. Suppose $[(e, a)] = [(f, b)]$

$$\begin{tikzcd}
e \bullet \arrow[loop right, "a"] \\
f \bullet \arrow[u, bend left, "(u,u\phi)"] \arrow[u, bend right, "(v,v\phi)"'] \arrow[loop right, "b"]
\end{tikzcd}$$

For some $u, v \in \mathbb{N}(Q)$, let

$$\begin{aligned}
(u, u\phi)(e, a)(v, v\phi) &= (f, b) \\
(uev, (u\phi \lhd e \lhd v) + a \lhd v + v\phi &= (f, b)
\end{aligned}$$

and so u starts at f and ends at e but $u \in \mathbb{N}(Q)$ and so $e = f$, and $u = v^{-1}$. Moreover v acts on $\mathcal{A}$ with $\phi : \mathbb{F}(Q) \to Q$ and since $v \in \mathbb{N}(Q)$, v acts trivially. Hence

$$\begin{aligned}
(u\phi \lhd e \lhd v) + a \lhd v + v\phi &= u\phi + a + v\phi \\
&= u\phi + a - u\phi = a
\end{aligned}$$

and so $a = b$. Therefore ι is injective.

We have that

$$\begin{aligned}
\ker(\pi_\phi) &= \{(w, a) : w \in \mathbb{N}(Q)\} \\
&= \{[(w, w\phi)(w\mathbf{r}, (w\phi)^{-1}a)]\} \\
&= \{[(w\mathbf{r}, (w\phi)^{-1}a)]\} \\
&\subseteq \mathrm{im}(\iota)
\end{aligned}$$

Therefore $\mathscr{E}_\phi$ is an extension as desired. $\square$

We denote $S \mathbin{/\!\!/} M_\phi$ by G_ϕ, and we denote the image in G_ϕ of an element $(w, a) \in S$ by $[w, a]_\phi$. Note that $[w, a]_\phi \pi_\phi = w\pi$. For the trivial homomorphism

$\tau \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$, we have $G_\tau = Q \ltimes \mathcal{A}$.

Let $\mathscr{E}$ be the set of extensions constructed from the normal ordered subgroupoids M_ϕ of S for $\phi \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ occurring in Lemma 5.3.6. Then the (abelian) group $\mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ acts transitively on $\mathscr{E}$: for $\alpha \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ we define $\mathscr{E}_\phi \triangleleft \alpha = \mathscr{E}_{\phi+\alpha}$.

Since $\mathbb{N}(Q)$ acts trivially on $\mathcal{A}$, when restricted to $\mathbb{N}(Q)$ any derivation $\delta : \mathbb{F}(Q) \to \mathcal{A}$ restricts to a morphism in $\mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$. This gives us the mapping

$$\rho : \mathrm{Der}(\mathbb{F}(Q), \mathcal{A}) \to \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$$

in the five-term exact sequence

$$0 \to \mathrm{Der}(Q, \mathcal{A}) \to \mathrm{Der}_\pi(\mathbb{F}(Q), \mathcal{A}) \to \mathrm{Hom}_Q(\mathbb{N}^{ab}(Q), \mathcal{A}) \to H^2(Q^I, \mathcal{A}^0) \xrightarrow{\varpi} H^2(\mathbb{F}(Q^I), \mathcal{A}^0)$$

using the fact that $\mathrm{Hom}_Q(\mathbb{N}^{ab}(Q), \mathcal{A}) = \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$. The key fact is

Proposition 5.3.7. *The extensions $\mathscr{E}_\phi$ and $\mathscr{E}_\psi$ are equivalent if and only if $-\phi+\psi : \mathbb{N}(Q) \to \mathcal{A}$ is the restriction of an ordered derivation $\mathbb{F}(Q) \to \mathcal{A}$.*

Proof. Suppose that the extensions $\mathscr{E}_\phi$ and $\mathscr{E}_\psi$ are equivalent: that is, there exist an ordered functor $\mu : G_\phi \to G_\psi$ making the diagram

$$\begin{tikzcd} & G_\phi \arrow[dd, "\mu"] \arrow[dr, "\pi_\phi"] & \\ \mathcal{A} \arrow[ur] \arrow[dr] & & Q \\ & G_\psi \arrow[ur, "\pi_\psi"'] & \end{tikzcd}$$

commute. Then for all $a \in \mathcal{A}_e$ and $(w, b) \in S$ we have:

$$[e, a]_\phi \, \mu = [e, a]_\psi \,, \tag{5.3.1}$$

$$[w, b]_\phi \, \mu\pi_\psi = [w, b]_\phi \, \pi_\phi = w\pi = [w, b]_\psi \, \pi_\psi. \tag{5.3.2}$$

Therefore $[w,b]_{\psi}^{-1}[w,b]_{\phi}\,\mu \in \ker \pi_{\psi}$. Setting $b = w\mathbf{r}$, we deduce that

$$[w,wr]_{\psi}^{-1}[w,w\mathbf{r}]_{\phi}\,\mu = [w\mathbf{r},a]_{\psi} \tag{5.3.3}$$

for some $a \in \mathcal{A}_{w\mathbf{r}}$. Since the map $\mathcal{A} \to G_{\psi}$, $a \mapsto [a\mathbf{r},a]_{\psi}$ is injective, the element a determined by equation (5.3.3) is unique. Hence we get a mapping $\delta : \mathbb{F}(Q) \to \mathcal{A}$ given by $w \mapsto a$ as in equation (5.3.3), with $a \in \mathcal{A}_{w\mathbf{r}}$, and satisfies

$$[w\mathbf{r},w\delta]_{\psi} = [w,w\mathbf{r}]_{\psi}^{-1}[w,w\mathbf{r}]_{\phi}\,\mu \tag{5.3.4}$$

We show that δ is an ordered derivation. Now for $u,v \in \mathbb{F}(Q)$ such that the composition uv exist, we have

$$\begin{aligned}
[v\mathbf{r},(uv)\delta]_{\psi} &= [uv,v\mathbf{r}]_{\psi}^{-1}[uv,v\mathbf{r}]_{\phi}\,\mu \\
&= [v,v\mathbf{r}]_{\psi}^{-1}[u,u\mathbf{r}]_{\psi}^{-1}[u,u\mathbf{r}]_{\phi}\,\mu\,[v,v\mathbf{r}]_{\phi}\,\mu \\
&= [v,v\mathbf{r}]_{\psi}^{-1}[u,u\mathbf{r}]_{\psi}^{-1}[u,u\mathbf{r}]_{\phi}\,\mu\,[v,v\mathbf{r}]_{\psi}[v,v\mathbf{r}]_{\psi}^{-1}[v,v\mathbf{r}]_{\phi}\,\mu \\
&= [v,v\mathbf{r}]_{\psi}^{-1}[u\mathbf{r},u\delta]_{\psi}[v,v\mathbf{r}]_{\psi}[v\mathbf{r},v\delta]_{\psi} \qquad \text{by (5.3.4)} \\
&= [v\mathbf{r},(u\delta)\triangleleft v + v\delta]
\end{aligned}$$

and therefore $(uv)\delta = (u\delta)\triangleleft v + v\delta$. Hence δ is an ordered derivation $\mathbb{F}(Q) \to \mathcal{A}$. It suffices to show that $(-\phi+\psi)$ is a restriction of δ.

Suppose that $w \in \mathbb{N}(Q)$. Then recalling that $w\mathbf{d} = w\mathbf{r}$ we have

$$[w,w\mathbf{r}]_{\phi} = \left[(w,w\mathbf{r})(w^{-1},w^{-1}\phi)\right]_{\phi} = \left[w\mathbf{r},w^{-1}\phi\right]_{\phi} \tag{5.3.5}$$

and so

$$\begin{aligned}
[w\mathbf{r},w\delta]_{\psi} &= \left[w\mathbf{d},w^{-1}\psi\right]_{\psi}^{-1}\left[w\mathbf{r},w^{-1}\phi\right]_{\phi}\mu \\
&= [w\mathbf{r},w\psi]_{\psi}\left[w\mathbf{r},w^{-1}\phi\right]_{\phi}\mu \\
&= [w\mathbf{r},w\psi]_{\psi}\left[w\mathbf{r},w^{-1}\phi\right]_{\psi}
\end{aligned}$$

and using equation (5.3.1) gives

$$[w\mathbf{r}, w\delta]_\psi \quad = \quad [w\mathbf{r}, w\psi - w\phi]$$

and, since $a \mapsto [a\mathbf{r}, a]_\psi$ is injective, we deduce that $w\delta = w(-\phi + \psi)$. Therefore if the extensions $\mathscr{E}_\phi$ and $\mathscr{E}_\psi$ are equivalent then $(-\phi + \psi)$ is the restriction of an ordered derivation $\delta : \mathbb{F}(Q) \to \mathcal{A}$.

For the converse, let $\eta : \mathbb{F}(Q) \to \mathcal{A}$ be an ordered derivation and $\phi \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$. Since η restricts to an $\mathbb{F}(Q)$–map $\mathbb{N}(Q) \to \mathcal{A}$ and $\mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ is closed under pointwise addition, the sum $\phi + \eta \in \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$. Set $\psi = \phi + \eta$ and define a mapping $\nu : G_\phi \to G_\psi$ by $[w, b]_\phi \mapsto [w, w\eta + b]_\psi$. Then for $u, v \in \mathbb{N}(Q)$ we have

$$(u, u\phi)(w, b)(v, v\phi) = (uw, (u\phi) \lhd w + b)(v, v\phi) = (uwv, (u\phi) \lhd w + b + v\phi)$$

since $\mathbb{N}(Q)$ acts trivially on $\mathcal{A}$, and so

$$\begin{aligned}
[(u, u\phi)(w, b)(v, v\phi)]_\phi \, \nu &= [uwv, (u\phi) \lhd w + b + v\phi]_\phi \, \nu \\
&= [uwv, (uwv)\eta + (u\phi) \lhd w + b + v\phi]_\psi \\
&= [uwv, (u\eta) \lhd wv + (wv)\eta + (u\phi) \lhd w + b + v\phi]_\psi \\
&= [uwv, (u\eta) \lhd w + w\eta + v\eta + (u\phi) \lhd w + b + v\phi]_\psi \\
&= [uwv, (u\psi) \lhd w + v\psi + w\eta + b]_\psi \\
&= [(u, u\psi)(w, w\eta + b)(v, v\psi)]_\psi \\
&= [w, w\eta + n]_\psi = [w, b]_\phi \, \nu
\end{aligned}$$

and so ν is well-defined. It is then easy to see that ν is an ordered functor, and that

$$\begin{tikzcd}
& G_\phi \arrow[dd, "\nu"] \arrow[dr, "\pi_\phi"] & \\
\mathcal{A} \arrow[ur] \arrow[dr] & & Q \\
& G_\psi \arrow[ur, "\pi_\psi"'] &
\end{tikzcd}$$

commutes, so that $\mathscr{E}_\phi$ and $\mathscr{E}_\psi$ are equivalent. $\square$

Corollary 5.3.8. *The set of equivalence classes of extensions of $\mathcal{A}$ by Q in $\mathscr{E}$ is in one-to-one correspondence with the cokernel of the restriction map*

$$\rho : \mathrm{Der}(\mathbb{F}(Q), \mathcal{A}) \to \boldsymbol{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A}) .$$

Proposition 5.3.9. *An extension $\mathcal{A} \xrightarrow{\iota} G \xrightarrow{\varphi} Q$ is equivalent to an extension $\mathscr{E}_\alpha$ for some $\alpha \in \boldsymbol{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$.*

Proof. Lift the quotient map π to $\pi_* : \mathbb{F}(Q) \to G$ making the diagram

$$\begin{tikzcd} & \mathbb{N}(Q) \arrow[r] & \mathbb{F}(Q) \arrow[d, "\pi_*"] \arrow[dr, "\pi"] & \\ & & & Q \\ \mathcal{A} \arrow[rr, "\iota"] & & G \arrow[ur, "\varphi"] & \end{tikzcd}$$

commute. Then $\mathbb{N}(Q)\pi_* \subseteq \mathcal{A}$ and we may define $\xi : \mathbb{F}(Q) \ltimes \mathcal{A} \to G$ by $(w, a)\xi = (w\pi_*)(a\iota)$ and

$$\ker \xi = \{(w, w^{-1}\pi_*) : w \in \mathbb{N}(Q)\} .$$

Since each $\mathcal{A}_e$ is abelian, the map $\alpha : w \mapsto w^{-1}\pi_*$ is a homomorphism $\mathbb{N}(Q) \to \mathcal{A}$, and $\mathscr{E}$ is equivalent to $\mathscr{E}_\alpha$. $\square$

In [33], Matthews discusses the concept of factor sets for modules over ordered groupoids. We define an *n–staircase* over an ordered groupoid G as an n-tuple $(g_1, \cdots, g_n)$ of morphisms of G where $g_i\mathbf{r} \leqslant g_{i+1}\mathbf{d}$. The set of n-staircases over G is denoted by $S_n(G)$. Suppose $\mathcal{A}$ is a module over G. Then a factor set is a function $\zeta : S_2(G) \to \mathcal{A}$ such that

FS1 for all $(h, g) \in S_2(G)$, $(h, g)\zeta \in \mathcal{A}_{hh^{-1}}$,

FS2 if $(k, h, g) \in S_3(G)$, then $k((h, g)\zeta|k^{-1}k) + (k, (h * g))\zeta = ((k * h), g)\zeta + (k, h)\zeta$.

Proposition 5.3.10. [33, Proposition 10.17] *Consider the extension $\mathcal{A} \xrightarrow{\iota} G \xrightarrow{\sigma} Q$. If we choose a transversal of the $\rho : Q \to G$. Then the function $\zeta : S_2(Q) \to \mathcal{A}$ defined by $\iota(\zeta(h, g)) = \rho\lfloor g\rfloor \cdot \rho\lfloor h\rfloor \cdot \rho\lfloor gh\rfloor^{-1}$ is a factor set.*

Remark 5.3.1 We note that we can recover the factor set discussed by Matthew from our construction. The factor set in the above proposition from [33] is essentially the restriction of the map $\pi_* : \mathbb{F}(Q) \to G$ to $\mathbb{N}(Q)$.

5.3.2 Main Result

Theorem 5.3.11. *The set of equivalence classes of extensions of $\mathcal{A}$ by Q is in one-to-one correspondence with the second cohomology group $H^2(Q^I, A^0)$.*

Proof. Consider the exact sequence $\mathbb{N}(Q) \to \mathbb{F}(Q) \xrightarrow{\pi} Q$ of ordered groupoids. By Lemma 5.1.7 we obtain the exact sequence of Q–modules $\mathbb{N}^{ab}(Q) \to D_\pi \to KQ$ which generates the five–term exact sequence

$$0 \to \mathrm{Der}(Q, \mathcal{A}) \to \mathrm{Der}_\pi(\mathbb{F}(Q), \mathcal{A}) \to \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A}) \to H^2(Q^I, \mathcal{A}^0) \xrightarrow{\varpi} H^2(\mathbb{F}(Q^I), \mathcal{A}^0)$$

by Theorem 5.1.9 and using the fact that $\mathrm{Hom}_Q(\mathbb{N}^{ab}(Q), \mathcal{A}) = \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$. Denote by $K\mathbb{F}(Q^I)$ the augmentation ideal of the epimorphism $\mathbb{Z}\mathbb{F}(Q^I) \to \Delta\mathbb{Z}$. Then a Q^I–map $K\mathbb{F}(Q^I) \to \mathcal{B}$ for $\mathcal{B} \in \mathrm{Mod}_{Q^I}$ corresponds to a homomorphism $\mathbb{F}(Q^I) \to \mathbb{F}(Q^I) \ltimes \mathcal{B}$. Now if $\mathcal{A} \to \mathcal{B}$ is some epimorphism of Q^I–modules then we obtain a lift

$$\begin{tikzcd} & \mathbb{F}(Q^I) \arrow[dl, dotted] \arrow[d] \\ \mathbb{F}(Q^I) \ltimes \mathcal{A} \arrow[r] & \mathbb{F}(Q^I) \ltimes \mathcal{B} \end{tikzcd}$$

via the freeness of $\mathbb{F}(Q^I)$. And so we get the corresponding lift $K\mathbb{F}(Q^I) \to \mathcal{A}$. Thus $K\mathbb{F}(Q^I)$ is projective and hence the sequence $K\mathbb{F}(Q^I) \to \mathbb{Z}\mathbb{F}(Q^I) \to \Delta\mathbb{Z}$ is a projective resolution of $\Delta\mathbb{Z}$. Applying the cohomological functor $H^n(-, \mathcal{A}^0)$ we get that $H^n(\mathbb{F}(Q^I), \mathcal{A}^0) = 0$ for $n > 0$. Thus the five term exact sequence above now reads

$$0 \to \mathrm{Der}(Q, \mathcal{A}) \to \mathrm{Der}_\pi(\mathbb{F}(Q), \mathcal{A}) \to \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A}) \xrightarrow{\varpi} H^2(Q^I, \mathcal{A}^0) \to 0\ .$$

Therefore ϖ is a bijection from the cokernel of the restriction map $\mathrm{Der}_\pi(\mathbb{F}(Q), \mathcal{A}) \to \mathbf{OGpd}_{\mathbb{F}(Q)}(\mathbb{N}(Q), \mathcal{A})$ to $H^2(Q^I, A^0)$ and thus by corollary 5.3.8 we get that the set of equivalence classes of extensions of $\mathcal{A}$ by Q is in one-to-one correspondences with $H^2(Q^I, \mathcal{A}^0)$ as desired. $\square$

Chapter 6

n-fold Extensions and Cohomology of Ordered Groupoids

This chapter is devoted to proving that the set of equivalence classes of *crossed n–fold* extensions of an abelian ordered groupoid $\mathcal{A}$ by an ordered groupoid Q is isomorphic to the cohomology group $H^{n+1}(Q^I, \mathcal{A}^0)$. The main result is that of Theorem 6.4.5. Our approach follows that of [23] by Huebschmann but at appropriate points we need to use constructions for ordered groupoids and ordered crossed complexes from chapter 5 of this thesis, and in particular the functor

$$\nabla : \mathcal{OCRS} \to \mathcal{OCHNS}$$

from section 5.1 and the identifications of cohomology groups from Theorem 3.4.4 and Theorem 5.3.11. We also supply some details omitted in [23], to verify that our ordered groupoid constructions do have the properties required in the argument. The main results in this chapter is a generalisation of the correspondence discussed in the previous chapter which is the ordered groupoid ananlogue of MacLane's result in [31] on the connection between the cohomology group $H^2(Q, \mathcal{A})$ and the set of extensions of an abelian group $\mathcal{A}$ by the group Q. We spend the first section to discuss the notion of n-fold extensions of ordered groupoids. In the second section we present some free constructions necessary for developing the connexion in the main results of this chapter. The third section is devoted to discussing homotopy of morphisms of ordered crossed complexes and finally present the main results in the fourth section.

6.1 Crossed n-fold extension

In this sections we will introduce the idea of crossed n-fold extensions of ordered groupoids. In the case of $n = 2$, the prefix is omitted and we say extensions of ordered groupoids. Our definitions will rely on the concept of ordered crossed complexes discussed in the previous chapter. The major motivation for the choice of crossed complexes as underlying concept is that crossed complexes are the natural generalisation of crossed modules which is a general theory for the theory of extensions in group theory.

Suppose C is an exact ordered crossed complex and Q is an ordered groupoid such that $Q \cong \pi_1(C)$, then the sequence C together with the quotient morphism $G \to Q$ whose kernel is $\mathrm{im}(\delta_1)$ is called an *ordered crossed resolution* of Q. Morphisms of ordered crossed resolutions of ordered groupoids are morphisms of ordered crossed complexes together with the induced map on the fundamental groupoids.

Definition A *crossed n–fold extension* of the ordered groupoid $\mathcal{A}$ by Q is an exact crossed complex

$$0 \to \mathcal{A} \xrightarrow{\gamma} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} G \xrightarrow{\varphi} Q \ .$$

It is evident that crossed n-fold extensions of ordered groupoids are special cases of ordered crossed resolutions of ordered groupoids.

Denote by $\mathscr{E}$ a crossed n-fold extension of $\mathcal{A}$ by Q. A morphism of crossed n-fold extensions $\mathscr{E} \to \mathscr{E}'$ is the pair (α, τ) where α is a morphism of crossed complexes and τ is the induced map on the fundamental groupoids. Morphisms of extensions give commutative diagrams of the form

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & G & \longrightarrow & Q \\ & & \downarrow{\scriptstyle \alpha_n} & & & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \tau} \\ 0 & \longrightarrow & \mathcal{A}' & \longrightarrow & \cdots & \longrightarrow & C_2' & \longrightarrow & G' & \longrightarrow & Q' \end{array}$$

Crossed n–fold extensions together with the corresponding morphisms form the category of crossed n-fold extensions. If we fix $\mathcal{A}$ and Q we obtain the subcategory of crossed n–fold extensions of $\mathcal{A}$ by Q. We shall later define an equivalence relation and an operation on this category that will give it an abelian group structure and show that it is isomorphic to the cohomology group $H^n(Q^I, \mathcal{A}^0)$. In the development we introduce the following concepts.

6.2 Free constructions

Constructions with universal properties are known to play pivotal roles in understanding some properties of many mathematical objects. A frequently visited example is the *free* group functor; a key tool in constructing the connexion between the set of equivalence classes of extensions and the second cohomology groups, see [22]. Often, free constructions in algebraic systems turns out to be left adjoint to some forgetful functors. For example, the free groupoid functor is left adjoint to the forgetful functor $F : \mathbf{Gpd} \to \mathrm{Grph}$ that associates a groupoid with its underlying directed graph (see [21] for details). The analogue for ordered groupoid is discussed in detail in the previous chapter. We will call a functor which is left adjoint to a forget functor a free functor. In what is to follow, we shall present appropriate descriptions of some forgetful functors and hence discuss the construction of free functors which are precisely left adjoints to the forgetful functors. The content of this section is an extension of the results in [6].

6.2.1 Free modules over ordered groupoids

We shall begin with a review of the concept of free modules over groupoids as presented in [6]. Let G be a groupoid. A G–module $\mathcal{B}$ is a totally disconnected abelian groupoid on G_0 together with a G–action. That is there is an abelian group $\mathcal{B}_e$ for every $e \in G_0$ and the G–action gives a group isomorphism $\lhd g : \mathcal{B}_e \to \mathcal{B}_f$ for $g \in G(e, f)$. The set up in the definition of G–modules comes a base point map $\mathcal{B} \to G_0$ and so every G–module comes with such a map as part of underlying structure for G–modules. Thus we describe the forgetful functor on

G–modules as follows.

Define the category of *sets over groupoids* denoted by $Sets/\mathbf{Gpd}$ as the category whose objects are maps $S \overset{\omega}{\to} G_0$ where S is a set and whose morphisms are pairs $(\sigma, \tau) : (S, G_0) \to (S', G'_0)$ so that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\omega} & G_0 \\ \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\tau} \\ S' & \xrightarrow{\omega'} & G'_0 \end{array}$$

commutes. We define composition of morphisms componentwise so that for another morphism $(\sigma', \tau') : (S', G'_0) \to (S'', G''_0)$ we get $(\sigma\sigma', \tau\tau') : (S, G_0) \to (S'', G''_0)$. We write $Sets/G$ for the subcategory of sets over a given groupoid G. In this case we fix $\tau = \mathrm{id} : G \to G$. It is clear that the category $Sets/\mathbf{Gpd}$ is the category of presheaf of sets over ordered groupoids.

We define the forgetful functor on the category of modules over groupoids as taking values in $Sets/\mathbf{Gpd}$. The forgetful functor forgets the abelian group structure on the underlying sets of the abelian groups in the family in a G–module, and the G–action.

Now we discuss the construction of modules over groupoids from objects of the category $Sets/\mathbf{Gpd}$. Suppose G is a groupoid and let S be a set together with some map $S \overset{\omega}{\to} G_0$. The G–module $\mathcal{B}^{\omega}$ defined over ω is presented as follows. $\mathcal{B}^{\omega}$ is a totally disconnected abelian groupoid on G_0 defined via the map ω. It is precisely a collection of abelian groups $\{\mathcal{B}^{\omega}_e\}_{e \in G_0}$. We define $\mathcal{B}^{\omega}_e$ as the free abelian group on the set

$$\{(s, g) \in (S, G) : g \in G((s)\omega, e)\}$$

else the zero abelian group whenever there is no such pairs satisfying the above condition. The G–action on $\mathcal{B}^{\omega}$ is presented as follows. Let (s, g) be a basis element of $\mathcal{B}^{\omega}_e$ and $h \in G(e, y)$, then

$$(s, g) \triangleleft h = (s, gh)$$

is a basis element in $\mathcal{B}^{\omega}_y$ and so we obtain an induced mapping $\mathcal{B}^{\omega}_e \to \mathcal{B}^{\omega}_y$.

Let $\mathcal{U}$ be the functor $\mathcal{U} : Sets/\mathbf{Gpd} \to \mathrm{Mod}_{\mathbf{Gpd}}$ defined by $(S \xrightarrow{\omega} G_0) \mapsto \mathcal{B}^{\omega}$. We state the following Proposition and we refer the reader to [5] for the detailed proof.

Proposition 6.2.1. [6, Proposition 7.3.1]*Suppose G is a groupoid with set of objects G_0. There is a forgetful functor $F : \mathrm{Mod}_{\boldsymbol{Gpd}} \to Sets/\boldsymbol{Gpd}$ which forgets the abelian structure on the family in a G–module and the G–action. This functor has a left adjoint $\mathcal{U} : Sets/\boldsymbol{Gpd} \to \mathrm{Mod}_{\boldsymbol{Gpd}}$ that gives the free module on a map $S \xrightarrow{\omega} G_0$ where S is a set.*

Thus $\mathcal{U}$ is a *free* module functor hence $\mathcal{B}^{\omega}$ is a free module. We proceed to discuss the idea of free modules over ordered groupoids.

Suppose G is an ordered groupoid with set of objects G_0. The natural order on G_0 makes G_0 a poset. Let P be a poset and let $\omega : P \to G_0$ be a morphism of posets. We define the category of posets over ordered groupoids written $\mathbf{P}/\mathbf{OGpd}$ as the category whose objects are morphisms $P \xrightarrow{\omega} G_0$ where P is a poset and whose morphisms are pairs $(\sigma, \tau) : (P, G) \to (P', G')$ where σ is a poset map and τ is an ordered morphisms of ordered groupoids satisfying the relation $\omega\tau = \sigma\omega'$. The subcategory $\mathbf{P}/G$ is the category of posets over a given ordered groupoid G. The objects of $\mathbf{P}/G$ are morphisms $\mathbf{P} \xrightarrow{\omega} G_0$ and morphisms are pairs (σ, id_G) where $\sigma : P \to P'$ such that $\omega = \sigma\omega'$. Suppose $(\sigma', \mathrm{id}_G) : (P', G) \to (P'', G)$ then the composition of (σ, id_G) and (σ', id_G) is given as $(\sigma\sigma', \mathrm{id}_G) : (P, G) \to (P'', G)$ such that $\omega = \sigma\sigma'\omega''$. In defining a module $\mathcal{B}$ over an ordered groupoid we obtain a natural ordered morphism $\mathcal{B} \to G_0$.

We say that the forgetful functor on modules over ordered groupoids is the functor $F : \mathrm{Mod}_{\mathbf{OGpd}} \to \mathbf{P}/\mathbf{OGpd}$ which forgets the abelian structure on the family in the G–module, and the G-action, but retains the ordering.

Now we discuss the construction of G–modules from objects of $\mathbf{P}/\mathbf{OGpd}$. Let P be a poset and G an ordered groupoid. Suppose that $P \xrightarrow{\omega} G_0$ is a morphism of posets. Then we construct the G–module $\mathcal{B}^{\omega}$ as follows. $\mathcal{B}^{\omega}$ is the family of abelian groups $\mathcal{B}^{\omega}_e$ indexed by G_0. The group $\mathcal{B}^{\omega}_e$ is defined as the free abelian group on the set

$$\{(p,g) \in (P,G) : g\mathbf{d} \leqslant (p)\omega \text{ and } g\mathbf{r} = e\}$$

otherwise the zero abelian group whenever there is not such pairs satisfying the above condition.

The action of G on $\mathcal{B}^\omega$ is described as follows. Let $h \in G(e,y)$ and $\sum n_{(p,g)}(p,g) \in \mathcal{B}^\omega_e$. Then

$$\sum n_{(p,g)}(p,g) \triangleleft h = \sum n_{(p,gh)}(p,gh) \in \mathcal{B}^\omega_y \; .$$

Now if $f \leqslant e$ in G_0 and let (p,g) be a basis element in $\mathcal{B}^\omega_e$. Then we obtain a unique corestriction $(f|g)$ of g in G such that $(f|g)\mathbf{d} \leqslant g\mathbf{d} \leqslant (p)\omega$ and $(f|g)\mathbf{r} = f$.

$$\begin{tikzcd}
\bullet^{(p)\omega} \arrow[d, dotted] & \\
\bullet \arrow[r, "g"] \arrow[d, dotted] & \bullet^{e} \arrow[d, dotted] \\
\bullet \arrow[r, "(f|g)"] & \bullet^{f}
\end{tikzcd}$$

Thus $(p,(f|g))$ is a basis element of $\mathcal{B}^\omega_f$ and so we obtain an induced map $\mathcal{B}^\omega_e \to \mathcal{B}^\omega_f$. Define the functor $\mathcal{U} : \mathbf{P/OGpd} \to \mathrm{Mod}_{\mathbf{OGpd}}$ by $(P \overset{\omega}{\to} G) \mapsto \mathcal{B}^\omega$. We show that $\mathcal{U}$ is a free functor by showing that it is left adjoint to the forgetful functor on modules over ordered groupoids in the following Proposition.

Proposition 6.2.2. *Suppose G is an ordered groupoid with object set G_0 and let F be the forgetful functor $\mathrm{Mod}_{\boldsymbol{OGpd}} \to \boldsymbol{P/OGpd}$ which forgets the abelian structure on the family in a G–module, and the G–action. Then the functor $\mathcal{U} : \boldsymbol{P/OGpd} \to \mathrm{Mod}_{\boldsymbol{OGpd}}$ by $(P \overset{\omega}{\to} G) \mapsto B^\omega$ is left adjoint to F.*

Proof. The goal of the proof is to show that for an ordered groupoid G with object set G_0, a morphism $P \overset{\omega}{\to} G_0$ of posets and a G–module $\mathcal{A}$ there is a canonical bijection $\mathbf{P/OGpd}(P \overset{\omega}{\to} G_0, (\mathcal{A})F) \cong \mathrm{Mod}_{\mathbf{OGpd}}(\mathcal{B}^\omega, \mathcal{A})$.
We note that $\mathcal{A}$ is a poset of abelian groups $\mathcal{A}_e$ indexed by G_0 and so $F(\mathcal{A})$ is a poset $\tilde{\mathcal{A}}$ of sets $\{\tilde{\mathcal{A}}_e\}_{e \in G_0}$. The definition gives a map $\tilde{\mathcal{A}} \overset{\lambda}{\to} G_0$. Let $\phi : (\mathcal{A})F \to (P \overset{\omega}{\to} G_0)$

be a $(\mathbf{P}/G)$–map. Then ϕ is a poset map $\tilde{\mathcal{A}} \to P$ such that $\phi\lambda = \omega$. It is a collection of maps $\{\phi_e\}_{e\in G_0} : \tilde{\mathcal{A}}_e \to P$ such that $\phi_e\omega = \lambda$. In particular triangles of the form

$$\begin{tikzcd} a \arrow[rr, "\phi_e"] \arrow[dr, "\lambda"'] & & p \arrow[dl, "\omega"] \\ & e & \end{tikzcd}$$

commutes for $p = a\phi_e$ where $a \in \tilde{\mathcal{A}}_e,\ p \in P$ and $e \in G_0$.

A G-module map $\psi : \mathcal{A} \to \mathcal{B}^\omega$ is necessarily a family of maps $\psi_e : \mathcal{A}_e \to \mathcal{B}^\omega_e$ which commutes with the G–actions. We define $\psi_e : \mathcal{A}_e \to \mathcal{B}^\omega_e$ by $a \mapsto (a\phi_e, e)$ for elements $a \in \mathcal{A}_e$ and $(a\phi, e) \in \mathcal{B}^\omega_e$. It is evident that ψ_e is determined by ϕ_e. Thus we obtain an injection

$$\rho : \mathrm{Mod}_{\mathbf{OGpd}}(\mathcal{B}^\omega, \mathcal{A}) \to \mathbf{P/OGpd}(P \xrightarrow{\omega} G_0, (\mathcal{A})F)\ .$$

We define the map $\bar{\rho} : \mathbf{P}/\mathsf{OGPD}(P \xrightarrow{\omega} G_0, (\mathcal{A})F) \to \mathrm{Mod}_{\mathbf{OGpd}}(\mathcal{B}^\omega, \mathcal{A})$ by $\phi_e \mapsto \psi_e$. Then we get that $\rho\bar{\rho}$ and $\bar{\rho}\rho$ are identities and so ρ is an isomorphism $\mathbf{P/OGpd}(P \xrightarrow{\omega} G_0, (\mathcal{A})F) \cong \mathrm{Mod}_{\mathbf{OGpd}}(\mathcal{B}^\omega, \mathcal{A})$. Therefore $\mathcal{U}$ is left adjoint to the forgetful functor F as desired. □

Therefore the module $\mathcal{B}^\omega$ is a free module over an ordered groupoid.

6.2.2 Free crossed modules over ordered groupoids

The goal of the sequel is to discuss the idea of *free ordered crossed modules.* The idea of freeness of crossed modules over groups is a widely known concept with many applications such as in [32] and [23]. We refer the reader to [44], [39] and [4] for details of the construction. In the case of unordered groupoids, the set up is presented in detail in [6]. We set out to discuss the appropriate concept of free ordered crossed modules with inspiration from [6].

Recall that an ordered crossed module is made up of an equivariant ordered morphism $N \xrightarrow{\mu} G$ of ordered groupoids whose restriction to the set of objects is the identity map, with N a totally disconnected ordered groupoid consisting of its vertex groups and a G–action on N. So μ is given as a family of group morphisms

$\mu_e : N_e \to G_e$ for $e \in G_0$. A key observation is that N comes with a poset structure and the image of the morphism μ given by $\mu_e : N_e \to G_e$ is contained in the local groups at $e \in G_0$. We write **P/OGpd** for the category of posets over ordered groupoids whose objects are ordered morphisms $P \xrightarrow{\omega} G$ such that the image of ω is contained in the local groups at identities of the ordered groupoid G and P is a poset, and whose morphisms are pairs $(\sigma, \tau) : (P, G) \to (P', G')$ where σ is a poset-map and τ is an ordered morphism of ordered groupoids so that $\omega\tau = \sigma\omega'$. We define composition of morphisms componenetwise and so given the morphism $(\sigma', \tau') : (P', G') \to (P'', G'')$ we have that $(\sigma\sigma', \tau\tau') : (P, G) \to (P'', G'')$.

6.2.2.1 Posets over ordered groupoids from ordered crossed modules.

It is evident that objects of the category **P/OGpd** are key components in structure of crossed modules. So we say the forgetful functor $F : \mathrm{Mod}_{\mathbf{OGpd}} \to \mathbf{P/OGpd}$ forgets the group structures on the family N and the G–action in the crossed module $N \xrightarrow{\mu} G$. Thus the forgetful functor associates every ordered crossed module $N \xrightarrow{\mu} G$ with the object $\tilde{N} \xrightarrow{\omega} G$ in $\mathbf{P}/G$ the category of posets over the ordered groupoid G.

6.2.2.2 Ordered crossed modules from posets over ordered groupoids.

Now we define a functor $\mathcal{U} : \mathbf{P/OGpd} \to \mathrm{Mod}_{\mathbf{OGpd}}$ which we will show that it is left adjoint to the forgetful functor F. Let $\tilde{N}$ be a poset together with an ordered morphism $\tilde{N} \xrightarrow{\omega} G$ such that the image of ω is contained in the local groups at identities in G. Then $\tilde{N} \xrightarrow{\omega} G$ is an object of **P/OGpd**. We construct a crossed module on ω as follows.

We define N^ω as a family of groups N^ω_e indexed by G_0. The group N^ω_e is the free group on pairs $(n, g) \in (\tilde{N}, G)$ such that $((n)\omega)\mathbf{d} \geqslant g\mathbf{d}$ and $g\mathbf{r} = e$, and otherwise the trivial group whenever no such pairs exist. To explain a G–action on N^ω, let (n, g) be a basis element in N^ω_e and let $h \in G(e, y)$. Then

$$(n, g) \triangleleft h = (n, gh) \in N^\omega_{h^{-1}h}$$

whenever gh is defined in G. This gives a map $N_e^{\omega} \to N_{h^{-1}h}^{\omega}$. If $e \geqslant f$ and (n,g) is a basis element in N_e^{ω}, then the unique corestriction of g to f is written $(f|g)$ such that $(f|g)\mathbf{d} \leqslant g\mathbf{d} \leqslant ((n)\omega)\mathbf{d}$ and $(f|g)\mathbf{r} = f$. So $(n,(f|g))$ is a basis element in N_f^{ω} and hence induces a mapping $N_e^{\omega} \to N_f^{\omega}$.

Now we need an ordered morphism $N^{\omega} \to G$ given by $N_e^{\omega} \to G_e$. Let $\theta : N^{\omega} \to G$ be an ordered morphism defined by

$$(n,g)\theta = g^{-1} * (n)\omega * g \in G_e$$

on a basis element (n,g) in N_e^{ω}. This extends to $\theta : N_e^{\omega} \to G_e$. We have that

$$\begin{aligned}
((n,g) \triangleleft h)\theta &= (n,gh)\theta \\
&= (gh)^{-1} * (n)\omega * gh \\
&= h^{-1} * g^{-1} * (n)\omega * g * h \\
&= h^{-1} * (n,g)\theta * h \\
&= h^{-1} \cdot (n,g)\theta \cdot h
\end{aligned}$$

Thus θ is an equivariant map and so $N^{\omega} \xrightarrow{\theta} G$ is a *precrossed* module. For the Peiffer relation, we want $(n,g) \triangleleft (m,h)\theta = (m,h)^{-1}(n,g)(m,h)$ for all $(m,h),(n,g) \in N_e^{\omega}$. And so for $e \in G_0$, we impose relations

$$(n, g * h^{-1} * (m)\omega * h) = (m,h)^{-1}(m,g)(m,h)$$

on N_e^{ω} for all $(n,g),(m,h) \in N_e^{\omega}$ to get the ordered crossed module $N^{\omega} \xrightarrow{\theta} G$ over ω.

Now define the functor $\mathcal{U} : \mathbf{P/OGpd} \to \mathrm{Mod}_{\mathbf{OGpd}}$ by

$$(\tilde{N} \xrightarrow{\omega} G) \mapsto (N^{\omega} \xrightarrow{\theta} G)\ .$$

6.2.2.3 Adjunction of functors

We spend the following paragraphs to show that the functor $\mathcal{U}$ is left adjoint to the forgetful functor on ordered crossed modules. The adjunction is depicted in the proposition below.

Proposition 6.2.3. *Let $F : \mathrm{Mod}_{\boldsymbol{OGpd}} \to \boldsymbol{P}/\boldsymbol{OGpd}$ be the forgetful functor which forgets the algebraic structures on the family N in an ordered crossed module $N \xrightarrow{\mu} G$ and the G–action. Then $\mathcal{U} : \boldsymbol{P}/\boldsymbol{OGpd} \to \mathrm{Mod}_{\boldsymbol{OGpd}}$ defined by*

$$(\tilde{N} \xrightarrow{\omega} G) \mapsto (N^{\omega} \xrightarrow{\theta} G)$$

is left adjoint to the forgetful functor F.

Proof. Let G be an ordered groupoid and let $\mathbf{P}/G$ be the category of posets over G. We denote the category of ordered crossed modules over G by $\mathcal{CM}_G$. Suppose $(\tilde{N} \xrightarrow{\omega} G) \in \mathbf{P}/G$ and $M \xrightarrow{\mu} G \in \mathcal{CM}_G$. It suffices to show that there is a canonical bijection

$$\mathbf{P}/G\left(\tilde{N} \xrightarrow{\omega} G, \tilde{M} \xrightarrow{\omega'} G\right) \cong \mathcal{CM}_G\left(N^{\omega} \xrightarrow{\theta} G, M \xrightarrow{\mu} G\right) .$$

Let $\phi : (\tilde{N} \xrightarrow{\omega} G) \to (\tilde{M} \xrightarrow{\omega'} G)$ be a $\mathbf{P}/G$–map. Then ϕ makes the triangle

$$\begin{tikzcd} \tilde{N} \arrow[rr, "\phi"] \arrow[dr, "\omega"'] & & \tilde{M} \arrow[dl, "\omega'"] \\ & G & \end{tikzcd}$$

commute. The map ϕ is necessarily the family of maps $\phi_e : \tilde{N}_e \to \tilde{M}_e$ giving rise to commutative triangles

$$\begin{tikzcd} \tilde{N}_e \arrow[rr, "\phi_e"] \arrow[dr, "\omega"'] & & \tilde{M}_e \arrow[dl, "\omega'"] \\ & G_e & \end{tikzcd}$$

Now define the $\mathcal{CM}_G$–map $\psi : (N^{\omega} \xrightarrow{\theta} G) \to (M \xrightarrow{\mu} G)$ by $\psi_e = \phi_e$ so that triangles of the form

$$\begin{tikzcd} N_e \arrow[rr, "\psi_e=\phi_e"] \arrow[dr, "\theta_e"'] & & M_e \arrow[dl, "\mu_e"] \\ & G_e & \end{tikzcd}$$

commute. Then we obtain the injection

$$\sigma : \mathcal{CM}_G\left(N^{\omega} \xrightarrow{\theta} G, M \xrightarrow{\mu} G\right) \to \mathbf{P}/G\left(\tilde{N} \xrightarrow{\omega} G, \tilde{M} \xrightarrow{\omega'} G\right)$$

since ψ is determined by ϕ. The map σ is invertible with inverse

$$\sigma^{-1} : \mathbf{P}/G\left(\tilde{N} \xrightarrow{\omega} G, \tilde{M} \xrightarrow{\omega'} G\right) \to \mathcal{CM}_G\left(N^{\omega} \xrightarrow{\theta} G, M \xrightarrow{\mu} G\right)$$

defined by $\phi_e \mapsto \psi_e$. We have that $(\psi)\sigma\sigma^{-1} = \psi$ and $(\phi)\sigma^{-1}\sigma = \phi$. Hence σ is a bijection and so $\mathcal{CM}_G\left(N^{\omega} \xrightarrow{\theta} G, M \xrightarrow{\mu} G\right) \to \mathbf{P}/G\left(\tilde{N} \xrightarrow{\omega} G, \tilde{M} \xrightarrow{\omega'} G\right)$ as desired. Therefore F is the left adjoint to the forgetful functor. $\square$

6.2.3 Free ordered crossed complexes

Definition An ordered crossed complex C is said to be of *free* type if

1. G is a free ordered groupoid,
2. $C_1 \to G$ is a free ordered crossed module; and
3. C_n are free Q-modules for $n \geqslant 3$.

We call an ordered crossed complex of free type a free ordered crossed complex. An exact free crossed complex with its fundamental groupoid isomorphic to a given ordered groupoid Q is called a *free ordered crossed resolution* of Q.

Proposition 6.2.4. *Every ordered groupoid admits a free ordered crossed resolution.*

Proof. Let Q be an ordered groupoid. We seek to construct a free ordered crossed complex C with its fundamental groupoid isomorphic to Q. We build up our resolution by induction. Let $\mathbb{F}(Q)$ be the free ordered groupoid on the underlying graph of Q as discussed in chapter 5. Then we obtain a canonical map $\varphi : \mathbb{F}(Q) \to Q$ defined by $\lfloor q \rfloor \mapsto q$. Denote by $\mathbb{N}(Q)$ the kernel of φ. Then $\mathbb{F}(Q)$ acts on $\mathbb{N}(Q)$ by conjugation and together with the injection $\iota : \mathbb{N}(Q) \to \mathbb{F}(Q)$ defines a crossed module. Let $\tilde{C}_2$ be the underlying poset of $\mathbb{N}(Q)$. Then we get an underlying morphism $\tilde{C}_2 \xrightarrow{\omega} \mathbb{F}(Q)$ such that the image of ω are the local groups at the identities of $\mathbb{F}(Q)$.

We form the free ordered crossed module $C_2 \to \mathbb{F}(Q)$ and choose a free Q–module C_3 mapping on to $\ker(C_2 \to \mathbb{F}(Q))$ giving $\delta_3 : C_3 \to C_2$. Then we choose a free Q–module C_4 mapping on to $\ker(\delta_3)$ and so on. Then we obtain the free ordered crossed resolution

$$\cdots \to \mathcal{C}_n \to \cdots \to C_3 \xrightarrow{\delta_3} C_2 \to \mathbb{F}(Q) \xrightarrow{\varphi} Q \,.$$

□

Proposition 6.2.5. *Suppose C is a free ordered crossed complex together with the quotient map $G \xrightarrow{\varphi} Q$ and C' an ordered crossed resolution of Q'. Then an ordered morphism $\tau : Q \to Q'$ may be lifted to a morphism $\alpha : C \to C'$ of the ordered crossed complexes.*

Proof. It suffices to show that there is a lift $\alpha : C \to C'$ of τ that makes the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}_n \arrow[r, "\delta_n"] \arrow[d, "\alpha_n"] & \cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "\alpha_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & G \arrow[r, "\varphi"] \arrow[d, "\alpha_1"] & Q \arrow[d, "\tau"] \\
\cdots \arrow[r] & \mathcal{C}'_n \arrow[r, "\delta'_n"] & \cdots \arrow[r] & C'_3 \arrow[r, "\delta'_3"] & C'_2 \arrow[r, "\delta'_2"] & G' \arrow[r, "\varphi'"] & Q'
\end{tikzcd}$$

commute. It is noted that G is a free ordered groupoid on the underlying graph of Q and the map φ is an epimorphism. So for every $g \in G$ if we select $g\alpha_1 \in G'$ such that $g\alpha_1\varphi' = g\varphi\tau$. Thus we obtain an ordered morphism $G \to G'$ following that φ' is an epimorphism. And so the first square on the right commutes.

Now $C_2 \xrightarrow{\delta_2} G$ is a free ordered crossed module on some map $P \to G$ where P is a poset. So we get that $p\delta_2\varphi\tau = p\delta_2\alpha_1\varphi' = \mathrm{id}_p$ for $p \in P$. Since the C' is a resolution of Q' we get that $p\delta_2\alpha_1 \subseteq \mathrm{im}(\delta'_2)$ and so we can select $p\alpha_2 \in C'_2$ such that $p\alpha_2\delta'_2 = p\delta_2\alpha_1$. This extends to a morphism $(\alpha_2, \alpha_1) : (C_2 \xrightarrow{\delta_2} G) \to (C'_2 \xrightarrow{\delta'_2} G')$ of crossed modules.

For the higher dimensions, the ordered groupoids C_n are free Q–modules and so are defined on some $P \xrightarrow{\omega} Q_0 \in \mathbf{P}/\mathsf{OGPD}$. We have that $p_{n+1}\delta_{n+1}\alpha_n\delta'_n = 0$. Since C' is a resolution of Q' we can select some $p'_{n+1} \in C'_{n+1}$ such that $p_{n+1}\delta_{n+1}\alpha_n = p'_{n+1}\delta'_{n+1}$. Define $p_{n+1}\alpha_{n+1} = p'_{n+1}$. By the freeness this extends to the morphisms required. Therefore τ lifts to the morphism $\alpha : C \to C'$ as required. □

A consequence of the above proposition is the following

Proposition 6.2.6. *Let $\mathscr{E}$ be a free n-fold extension by Q and $\mathscr{E}'$ an n-fold extension by Q'. Any ordered morphism $\tau : Q \to Q'$ may be lifted to a morphism (α, τ) of n-fold extensions $\mathscr{E} \to \mathscr{E}'$.*

6.3 Homotopy of morphisms of ordered crossed complexes

A concept that is well known in the study of chain complexes over groups is *homotopy* of morphisms of chain complexes. The analogous definition in groupoids involves extra care as a result of the amount of data contained in the nonabelian component of crossed complexes. A comprehensive discussion of an appropriate notion of homotopy of morphisms of crossed complexes is presented by Brown, Sivera and Higgins in [6] for unordered groupoids. We extend this to crossed complexes over ordered groupoids.

Definition Let $\alpha, \beta : C \to C'$ be ordered morphisms of the ordered crossed complexes. A *homotopy* h from α to β written $\alpha \simeq \beta$ is a family of order preserving maps $\{h_n\} : C_n \to C'_{n+1}$ presented as

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[dl, "h_{n+1}"'] & C_n \arrow[r, "\delta_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "\delta_{n-1}"'] \arrow[dl, "h_{n-1}"'] & \cdots \arrow[r, "\delta_3"] \arrow[dl, "h_{n-2}"'] & C_2 \arrow[r, "\delta_2"] \arrow[dl, "h_2"'] & G \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[dl, "h_1"'] & G_0 \arrow[dl, "h_0"'] \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\delta'_{n+1}"] & C'_n \arrow[r, "\delta'_n"] & C'_{n-1} \arrow[r, "\delta'_{n-1}"] & \cdots \arrow[r, "\delta'_3"] & C'_2 \arrow[r, "\delta'_2"] & G' \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & G_0
\end{tikzcd}$$

satisfying the following:

H01

$$h_n(g) = \begin{cases} (g)h_0 \in G'((g)\alpha_1, (g)\beta_1) & \text{if } g \in G_0 \\ (g)h_1 \in C'_2((f)\beta_2) & \text{if } g \in G(e, f) \\ (g)h_n \in C'_{n+1}((f)\beta_n) & \text{if } g \in C_n((f)\beta) n \geqslant 2 \end{cases}$$

H02 the map $h_1 : G \to C'_2$ is a β_1-derivation so that whenever $g_1 g_2$ is defined in G then $(g_1 g_2)h_1 = g_1 h_1 \triangleleft g_2 \beta_1 + g_2 h_1$, H03 h_2 preserves actions G defined over β_1. Thus suppose $g \in G(e,f)$, $c_1 \in C_2(e)$ and $c_2 \in C_2(f)$ so that $c_1 \triangleleft g$ exists, then we have $(c_1 \triangleleft g + c_2)h_2 = (c_1 h_2) \triangleleft g\beta_1 + c_2 h_2$. Hence h_2 is a G-morphism. Also h_n for $n \geqslant 3$ is a linear map that is $c_1, c_2 \in C_n$ such that $c_1 + c_2$ exists, then $(c_1 + c_2)h_n = c_1 h_n + c_2 h_n$,

H04 For any diagram

$$\begin{tikzcd} & C_n \arrow[r, "\delta_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "\beta_n"'] \arrow[d, shift left, "\alpha_n"] & C_{n-1} \arrow[dl, "h_{n-1}"] \\ C'_{n+1} \arrow[r, "\delta'_{n+1}"] & C'_n & \end{tikzcd}$$

the difference between the vertical maps is evaluated as sum of the maps in triangles. We present this as follows

- if $g \in G(e,f)$ then

$$(g)\beta_1 = (g)\alpha_1 - (x)h_1\delta'_2 + (e)h_0 - (f)h_0 \ ,$$

- if $g \in C_n(f)$ for $n \geqslant 2$ then

$$(g)\beta = (g)\alpha \triangleleft fh_0 - g\delta_n h_{n-1} - gh_n\delta'_{n+1}$$

It is to be noted that the ordered morphisms α_0 and β_0 are not required to be the same.

Lemma 6.3.1. *Homotopy between morphisms of ordered crossed complexes is an equivalence relation.*

Proof. It is evident that the homotopy relation is symmetric and reflexive. For transitivity, let $h : \alpha \simeq \beta$ and $h' : \beta \simeq \gamma$ be homotopies of morphisms of ordered crossed complexes. We define the composite homotopy $H : \alpha \simeq \gamma$ by

$$(g)\mathrm{H}_n = \begin{cases} (g)h_0 + (g)h'_0 & \text{if } \ g \in G_0 \\ (g)h'_n + (g)h_n \triangleleft (f)h'_0 & \text{if } \ g \in G(e,f) \ \text{ or } \ C_n(f), n \geqslant 2 \end{cases}$$

$\square$

In the sequel, morphisms of ordered crossed complexes are identity on the object sets.

Proposition 6.3.2. *Let C be a free ordered crossed complex with fundamental groupoid Q and C' an ordered crossed resolution of Q'. The morphism $\alpha : C \to C'$ that induced by the morphism $Q \xrightarrow{\tau} Q'$ between the fundamental groupoids is unique up to homotopy equivalence.*

Proof. Let C be a free ordered crossed complex C with fundamental groupoid Q and C' be resolution of Q'. Suppose $\alpha, \beta : C \to C'$ are morphisms of crossed complexes such that they induce the same map $\tau : Q \to Q'$ in the fundamental groupoids. Then we show that there is a map h of degree 1 from C to C' presented as

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[dl, "h_{n+1}"'] & C_n \arrow[r, "\delta_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \arrow[r, "\delta_3"] \arrow[dl, "h_{n-2}"'] & C_2 \arrow[r, "\delta_2"] \arrow[dl, "h_2"'] & G \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[dl, "h_1"'] & G_0 \arrow[dl, "h_0"'] \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\delta'_{n+1}"] & C'_n \arrow[r, "\delta'_n"] & C'_{n-1} \arrow[r, "\delta'_{n-1}"] & \cdots \arrow[r, "\delta'_3"] & C'_2 \arrow[r, "\delta'_2"] & G' \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & G'_0
\end{tikzcd}$$

so that h is a homotopy from α to β. We have that $\beta_0 = \alpha_0$ is identity on G_0. Let G be the free ordered groupoid on the underlying graph $\Gamma(Q)$ of Q and let $g \in \Gamma(Q)$. Then $g\alpha_1\delta'_1 = g\delta_1\alpha_0 = g\beta_1\delta'_1 = g\delta_1\beta_0$. Thus $(g\alpha_1 - g\beta_1)\delta'_1 = e$ for some identity $e \in G_0$ the target of g. Since C' is a resolution one could find some $c'_2 \in C'_2$ such that $g\alpha_1 - g\beta_1 = c'_2\delta'_2$. This defines a map $\Gamma(Q) \to C'_2$ and by the freeness of G we obtain the induced map $h_1 : G \to C'_2$.

Now for $n \geqslant 2$ we have that $(c_n)\beta_n - (c_n)\alpha \lhd fh_0 - c_n\delta_n h_1 \in C'_n$ and that

$$\begin{aligned}
((c_n)\beta_n - (c_n)\alpha \lhd fh_0 - c_n\delta_n h_{n-1})\,\delta'_n &= (c_n)\beta_n\delta'_n - (c_n)\alpha \lhd fh_0\delta'_n + c_n\delta_n h_{n-1}\delta'_n \\
&= (c_n\delta_n)\beta_{n-1} - (c_n\delta_n)\alpha_{n-1} \lhd fh_0 - c_n\delta_n h_{n-1}\delta'_n
\end{aligned}$$

But

$$c_n\delta_n h_{n-1}\delta'_n = -(c_n\delta_n)\beta_{n-1} + (c_n\delta_n)\alpha_{n-1} \lhd fh_0 - c_n\delta_n\delta_{n-1}h_{n-2}\,.$$

Hence

$$((c_n)\beta_n - (c_n)\alpha \triangleleft fh_0 - c_n\delta_n h_{n-1})\,\delta'_n \quad = \quad -c_n\delta_n\delta_{n-1}h_{n-2} = 0$$

By the exactness of C' one can find some $y \in C'_{n+1}$ such that $y\delta'_{n+1} = (c_n)\beta_n - (c_n)\alpha_n \triangleleft fh_0 - c_n\delta_n h_{n-1}$. Therefore setting $c_n h_n = y$ we obtain the required homotopy h from α to β. □

Proposition 6.3.3. *Suppose C is a free n-fold extension and C' an n-fold extension. Let the ordered morphisms $\alpha, \beta : C \to C'$ have the same right end map. Then there is a homotopy $h : \alpha \simeq \beta$.*

It is clear that no two homotopy classes of ordered morphisms of ordered cross complexes induce the same right end map $Q \to Q'$ and so we make the following proposition.

Proposition 6.3.4. *There is an injection from* $\mathrm{Hom}(Q, Q')$ *to the set of homotopy classes of morphisms of crossed n-fold extensions with same right end maps.*

Corollary 6.3.5. *Free ordered crossed resolution of an ordered groupoid Q belong to the same homotopy class.*

One needs to consider the identity map $\mathrm{id}_Q : Q \to Q$ as the right end map and infer from Proposition 6.3.3 the required result.

Suppose C is a free ordered crossed resolution of the ordered groupoid Q. Let $\mathcal{J}_n$ denote the kernel of the map $C_{n-1} \to C_{n-2}$ and denote the sequence of ordered groupoids

$$0 \to \mathcal{J}_n \to C_{n-1} \to \cdots \to C_3 \to C_2 \to G \to Q$$

by C^n. The sequence is indeed a free ordered crossed n–fold extension of $\mathcal{J}_n$ by Q for $n > 1$.

Suppose $\mathscr{E}$ is an n–fold extension of an ordered groupoid $\mathcal{A}$ by Q. Then every morphism $Q \to Q$ can be lifted to a morphism of ordered crossed complexes $C^n \to \mathscr{E}$ which is unique up to homotopy equivalence from Proposition 6.3.3.

6.4 Main Result

In this section, we shall discuss the main result of this chapter. We commence with the discussion of some results that leads to the main result.

Suppose C is the a free ordered crossed resolution of the ordered groupoid Q presented as

$$\cdots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} G \xrightarrow{\varphi} Q\ .$$

The functor ∇ gives the ordered chain complex over Q

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_2^{ab} \xrightarrow{\partial_2} D_\varphi \xrightarrow{\partial_1} KQ$$

where D_φ and KQ are the ordered derived module over $G \xrightarrow{\varphi} Q$ and the augmentation ideal respectively. By Lemma 5.1.7, the ordered chain complex is a resolution of the augmentation ideal. Denote by $\widehat{C}$ the exact sequence of Q–modules

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_2^{ab} \xrightarrow{\partial_2} D_\varphi\ .$$

Suppose $\mathcal{A}$ is a Q–module. Applying the contravariant functor $\mathbf{OGpd}_Q(-,\mathcal{A})$ to the sequence $\widehat{C}$ produces the sequence

$$\mathbf{OGpd}_Q(D_\varphi,\mathcal{A}) \to \mathbf{OGpd}_Q(C_2^{ab},\mathcal{A}) \to \cdots \to \mathbf{OGpd}_Q(C_n,\mathcal{A}) \to \cdots\ .$$

The adjunction of the derived functor leads to the equivalence of the sequence $\mathbf{OGpd}_Q(\widehat{C},\mathcal{A})$ to the sequence $\mathbf{OGpd}_Q(C,\mathcal{A})$. An immediate consequence is that a measure of the inexactness of $\mathbf{OGpd}_Q(\widehat{C},\mathcal{A})$ is identified with the cohomology groups of $\mathbf{OGpd}_Q(C,\mathcal{A})$. The cohomology groups are characterised as follows.

Proposition 6.4.1. *Suppose C is a free ordered resolution of the ordered groupoid Q and let $\mathcal{A}$ be a Q–module. Then there are canonical bijections $H^0(Q,\mathcal{A}) = H^0(\boldsymbol{OGpd}_Q(C,\mathcal{A})) \cong \mathrm{Der}(Q,\mathcal{A})$ and $H^n(\boldsymbol{OGpd}_Q(C,\mathcal{A})) \cong H^{n+1}(Q^I,\mathcal{A}^0)$ for $n >$*

0.

Proof. Let C be a free resolution of the ordered groupoid Q and let $\mathcal{A}$ be a Q–module. We obtain the sequence $\mathbf{OGpd}_Q(\widehat{C}, \mathcal{A})$

$$\mathrm{Der}_\varphi(G, \mathcal{A}) \to \mathbf{OGpd}_Q(C_2^{ab}, \mathcal{A}) \to \cdots \to \mathbf{OGpd}_Q(C_n, \mathcal{A}) \to \cdots .$$

In dimension 0, $H^0(\widehat{C}, \mathcal{A})$ is the kernel of the map $\mathrm{Der}(D_\varphi, \mathcal{A}) \to \mathbf{OGpd}_Q(C_2^{ab}, \mathcal{A})$, that is the group of ordered derivations $D_\varphi \to \mathcal{A}$ whose restrictions to C_2^{ab} are zero. The Q-map $D_\varphi \to \mathcal{A}$ corresponds to a φ–derivation $G \to \mathcal{A}$ and so we get the commutative square

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & Q \arrow[d, "="] \\ Q \ltimes \mathcal{A} \arrow[r, "(\mathrm{id},0)"] & Q \end{tikzcd}$$

in $\mathcal{OG}^2$, and composing with $C_2 \to G$ gives the diagram

$$\begin{tikzcd} C_2 \arrow[r, "\delta_2"] \arrow[d] & G \arrow[r, "\varphi"] \arrow[d] & Q \arrow[d, "="] \\ C_2^{ab} \arrow[r, "0"] & Q \ltimes \mathcal{A} \arrow[r, "(\mathrm{id},0)"] & Q \end{tikzcd}$$

Recall that $Q \cong G/\operatorname{im}(\delta_2)$. Now the zero map $C_2^{ab} \to \mathcal{A}$ corresponds to derivations from $G \to \mathcal{A}$ that vanishes on the image of C_2 under δ_2. These induce derivations $Q \to A$. Thus $H^0(\mathbf{OGpd}_Q(C, \mathcal{A})) \cong \mathrm{Der}(Q, \mathcal{A})$.
By Theorem 3.4.4 and following Theorem 5.3.11 we obtain the isomorphisms

$$H^n(\mathbf{OGpd}_Q(C, \mathcal{A})) \cong H^{n+1}(Q^I, \mathcal{A}^0)$$

for $n > 0$ as desired. □

We now turn our attention to n–fold extensions.

Definition The crossed n-fold extensions $\mathscr{E}, \mathscr{E}'$ of $\mathcal{A}$ by Q are said to be related if there is a morphism of crossed n-fold extensions $(1, \tau, 1) : \mathscr{E} \to \mathscr{E}'$.

The morphism of extension is not necessarily an isomorphism hence the relation is not symmetric. So we pass to the smallest equivalence relation containing the relation and say that the extensions $\mathscr{E}$ and $\mathscr{E}'$ are in the same *similarity* class if there is a finite sequence of n-fold extensions $\mathscr{E} = \mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_k = \mathscr{E}'$ such that there exist morphisms $\mathscr{E}_j \to \mathscr{E}_{j+1}$ or $\mathscr{E}_{j+1} \to \mathscr{E}_j$. We denote the set similarity relation by $\equiv$ and the similarity class of $\mathscr{E}$ by $[\mathscr{E}]$. Let C be a free ordered crossed resolution of Q and $\mathscr{E}$ an extension of $\mathcal{A}$ by Q. The identity map on Q gets lifted to a morphism $(\nu, \alpha) : C \to \mathscr{E}$ of crossed complexes

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}_n \arrow[r] \arrow[d, "\nu"] & C_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\alpha_3"] & C_2 \arrow[r] \arrow[d, "\alpha_2"] & G \arrow[r, "\varphi"] \arrow[d, "\alpha_1"] & Q \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & \mathcal{A} \arrow[r] & A_{n-1} \arrow[r] & \cdots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & G' \arrow[r, "\varphi'"] & Q
\end{tikzcd}$$

From Proposition 6.4.1, $H^{n+1}(Q^I, \mathcal{A}^0)$ is given by $\ker(\mathbf{OGpd}_Q(C_n, \mathcal{A}) \to \mathbf{OGpd}_Q(C_{n+1}, \mathcal{A})$, but we have that

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[d] \\
0 \arrow[r] & \mathcal{A}
\end{tikzcd}$$

thus the elements of $\mathbf{OGpd}_Q(C_n, \mathcal{A})$ determine equivalence classes in the cohomology groups $H^{n+1}(Q^I, \mathcal{A}^0)$. Thus ν is a representative of the class $[\nu] \in H^{n+1}(Q^I, \mathcal{A}^0)$. Let C^n be the extension by Q obtained from the free ordered crossed resolution C of Q. Then the identity map on Q lifts to the morphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}_n \arrow[r] \arrow[d, "\nu"] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d] & G \arrow[r, "\pi"] \arrow[d] & Q \arrow[d, equal] \\
0 \arrow[r] & \mathcal{A} \arrow[r] & A_{n-1} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r] & G' \arrow[r] & Q
\end{tikzcd}$$

of n–fold extensions.

Now we discuss some construction which gives representatives of similarity classes of extensions by Q. Suppose we start with the data; a free n–fold extension C^n obtained from the free crossed resolution C of Q and a map $\nu : \mathcal{J}_n \to \mathcal{A}$ where $\mathcal{J}_n = \ker(C_{n-1} \to C_{n-2})$. We show that ν generates an n–fold extension of $\mathcal{A}$ by Q. Define $M_\nu = \{(k, k\nu) | k \in \mathcal{J}_n\} \subseteq C_{n-1} \times \mathcal{A}$. Denote by $C_{n-1,\nu}$ the quotient of

$C_{n-1} \times A$ by M_ν. Then the diagram

$$\begin{tikzcd} \mathcal{J}_n \arrow[r, "\iota"] \arrow[d, "\nu"'] & C_{n-1} \arrow[d, "\mu"] \\ \mathcal{A} \arrow[r, "\varrho"] & C_{n-1,\nu} \end{tikzcd}$$

is a pushout square. Hence we obtain a unique map $\delta'_{n-1} : C_{n-1,\nu} \to C_{n-2}$ such that $\mu\delta'_{n-1} = \delta_{n-1}$ and since $\iota\delta_{n-1} = \nu\varrho\delta'_{n-1}$, it implies $\varrho\delta'_{n-1} = 0$. Also the map $\delta_{n-1} : C_{n-1} \to C_{n-2}$ factors through δ'_{n-1} hence they have the same range.

Therefore the sequence

$$0 \to \mathcal{A} \xrightarrow{\varrho} C_{n-1,\nu} \xrightarrow{\delta'_{n-1}} C_{n-2} \to \cdots \to C_1 \to G \to Q$$

is a free crossed n-fold extension of $\mathcal{A}$ by Q and we denote it by νC^n. There is a morphism $(1, \beta, 1) : \nu C^n \to \mathcal{E}$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d, equal] & C_{n-1,\nu} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d] & G \arrow[r, "\pi"] \arrow[d] & Q \arrow[d, equal] \\ 0 \arrow[r] & \mathcal{A} \arrow[r] & A_{n-1} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r] & G' \arrow[r] & Q \end{tikzcd}$$

So for each $\nu \in \mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A})$ we can define an n-fold extension of $\mathcal{A}$ by Q which is a representative of some similarity class. This defines a map from $\mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A})$ to equivalence classes of n-fold extensions of $\mathcal{A}$ by Q. Thus the discussion above leads to the following proposition.

Proposition 6.4.2. *Each equivalence class of crossed n-fold extension of $\mathcal{A}$ by Q has a representative of the form νC^n*

We denote by $\mathscr{M}(Q, \mathcal{A})$ the set of equivalence classes of extensions of $\mathcal{A}$ by Q. Then we have a map from $\mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A}) \xrightarrow{\sigma} \mathscr{M}(Q, \mathcal{A})$ defined by $\nu \to [\nu C^n]$. The set of ordered functors $\mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A})$ is an abelian group under pointwise addition in $\mathcal{A}$. We show that σ is in fact a group homomorphism with respect to the *Baer sum* on $\mathscr{M}(Q, \mathcal{A})$.

The Baer sum of the extensions $\mathscr{E}$ and $\mathscr{E}'$ written $\widehat{\mathscr{E}} = \mathscr{E} * \mathscr{E}'$ is defined as the extension of $\mathcal{A}$ by Q presented as the sequence in the middle of the diagram

$$\begin{tikzcd}
 & A_{n-1} \arrow[r] & \cdots \arrow[r] & G \arrow[dr] & \\
\mathcal{A} \arrow[ur] \arrow[r] \arrow[dr] & \widehat{A}_{n-1} \arrow[r] & \cdots \arrow[r] & \widehat{G} \arrow[r] & Q \\
 & A'_{n-1} \arrow[r] & \cdots \arrow[r] & G' \arrow[ur] &
\end{tikzcd}$$

It is constructed as follows. The Q–module $\widehat{A}_{n-1} = (A/N)$ where A is the ordered subgroupoid $A = \{(a_{n-1}, a'_{n-1}) \in A_{n-1} \times A'_{n-1} | a_{n-1}\delta_{n-1} = a'_{n-1}\delta'_{n-1}\}$ of $A_{n-1} \times A'_{n-1}$ and $N = \{N_e\}_{e \in Q_0}$ where $N_e = \{(a\delta_n, -a\delta'_n) | a \in \mathcal{A}_e\}$. We have that $\widehat{A}_k = A_k \times A'_k$ for $2 \leqslant k \leqslant n-2$ and $\widehat{G}$ is the pullback of the morphisms $G \to Q$ and $G' \to Q$. It is noted that given the extension $(\mu+\nu)C^n$ and the Baer sum $\mu C^n * \nu C^n$ of μC^n and νC^n, the identity morphism on Q lifts to a morphism

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "\mathrm{id}"] & C_{n-1,(\mu+\nu)} \arrow[r] \arrow[d] & \mathcal{C}_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & G \arrow[r] \arrow[d] & Q \arrow[d, "\mathrm{id}"] \\
\mathcal{A} \arrow[r] & \widehat{C}_{n-1} \arrow[r] & \widehat{C}_{n-2} \arrow[r] & \cdots \arrow[r] & \widehat{G} \arrow[r] & Q
\end{tikzcd}$$

of n-fold extensions. Thus we have $(\mu+\nu)C^n \equiv \mu C^n * \nu C^n$. The Baer sum of extensions induces a sum on the similarity classes. Consequently the epimorphism $\mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A}) \to \mathscr{M}(Q, \mathcal{A})$ is a homomorphism of abelian groups with respect to the Baer sum. The zero element of $\mathscr{M}(Q, \mathcal{A})$ is the similarity class of $0C^n$ and the inverse of an extension $\mathscr{E}$

$$\mathcal{A} \xrightarrow{\gamma} C_{n-1} \to \cdots \to G \to Q$$

is the extension $-\mathscr{E}$

$$\mathcal{A} \xrightarrow{-\gamma} C_{n-1} \to \cdots \to G \to Q\ .$$

Again we consider a free ordered crossed resolution C of Q and let $\mathcal{J}_n = \ker(C_{n-1} \to C_{n-2})$. In particular $\mathcal{J}_1 = \ker(G \to Q)$ and $\mathcal{J}_2 = \ker(C_2 \to G)$. Then we have the following lemma.

Lemma 6.4.3. *Suppose that $\nu \in \boldsymbol{OGpd}_Q(\mathcal{J}_n, \mathcal{A})$ for $n \geqslant 1$ extends to a mapping $C_{n-1} \to \mathcal{A}$ that is*

1. *a derivation $G \to \mathcal{A}$ if $n = 1$*
2. *a G-map $C_1 \to \mathcal{A}$ if $n = 2$*
3. *a Q-map $C_{n-1} \to \mathcal{A}$ if $n \geqslant 3$*

Then there is a split exact sequence

$$\mathcal{A} \to C_{n-1,\nu} \to \mathcal{J}_{n-1} \ .$$

Proof. We have that $\mathcal{J}_{n-1} = \ker(C_{n-2} \to C_{n-3})$ is a quotient of C_{n-1} with a short exact sequence

$$\mathcal{J}_n \to C_{n-1} \to \mathcal{J}_{n-1} \ .$$

In the pushout square

$$\begin{tikzcd} \mathcal{J}_n \arrow[r] \arrow[d] & C_{n-1} \arrow[d] \\ \mathcal{A} \arrow[r] & C_{n-1,\nu} \end{tikzcd}$$

we map $C_{n-1} \to \mathcal{J}_{n-1}$ as above and $\mathcal{A} \to \mathcal{J}_{n-1}$ by the zero map. Thus we get an induced map from the pushout $C_{n-1,\nu} \to \mathcal{J}_{n-1}$ with kernel $\mathcal{A}$ and so the exact sequence

$$\mathcal{A} \to C_{n-1,\nu} \to \mathcal{J}_{n-1} \ .$$

Now since ν extends to a mapping $C_{n-1} \to \mathcal{A}$ we get an induced mapping $C_{n-1,\nu} \to \mathcal{A}$ which makes the exact sequence split on the left as desired. $\square$

Lemma 6.4.4. *The extensions μC^n and νC^n belong to the same similarity class if and only if $\mu - \nu$ is extensible over C_{n-1} to a map $C_{n-1} \to \mathcal{A}$.*

Proof. Suppose the extensions μC^n and νC^n belong to the same similarity class: that is, there are some crossed n-fold extensions $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \cdots, \mathscr{E}_k$ of $\mathcal{A}$ by Q where $\mathscr{E}_k = \nu C^n$ together with morphisms $(1, \tau^i, 1)$ as presented in the diagram below.

$$\begin{tikzcd} \mu C^n \arrow[dr, "{(1,\tau^1,1)}"] & & \mathscr{E}_2 \arrow[dl, "{(1,\tau^2,1)}"'] \arrow[dr, "{(1,\tau^3,1)}"] & & \mathscr{E}_4 \arrow[dl, "{(1,\tau^4,1)}"'] \arrow[dr, "{(1,\tau^5,1)}"] & & {} \arrow[dl, dotted] \arrow[dr, dotted] & & \mathscr{E}_k = \nu C^n \arrow[dl, "{(1,\tau^k,1)}"] \\ & \mathscr{E}_1 & & \mathscr{E}_3 & & \mathscr{E}_5 & & \mathscr{E}_{k-1} & \end{tikzcd}$$

We have chosen k to be even. Otherwise one could add an identity morphism at $\mathscr{E}_{k-1}$ to make it even. By default we can find morphisms $(\mu, \alpha^0, 1) : C^n \to \mu C^n$ presented as

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{J}_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & G & \longrightarrow & Q \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & C_{n-1,\mu} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & G' & \longrightarrow & Q
\end{array}$$

$(\nu, \alpha^k, 1) : C^n \to \nu C^n$ in the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{J}_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & G & \longrightarrow & Q \\
 & & \downarrow{\scriptstyle \nu} & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & C_{n-1,\nu} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & G' & \longrightarrow & Q
\end{array}$$

and $(\lambda^i, \alpha^i, 1) : C^n \to \mathscr{E}_k$

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{J}_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & G & \longrightarrow & Q \\
 & & \downarrow{\scriptstyle \lambda^i} & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathcal{A} & \longrightarrow & A_{n-1} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & G' & \longrightarrow & Q
\end{array}$$

arising as the lift of the identity map on Q following Proposition 6.2.5. These maps exist and are unique up to homotopy equivalence. Thus the compositions $(\mu, \alpha^0\tau^1, 1), (\lambda^2, \alpha^2\tau^2, 1) : C^n \to \mathscr{E}_1$ are homotopy equivalent by Proposition 6.3.2. Also $(\lambda^2, \alpha^2\tau^3, 1), (\lambda^4, \alpha^4\tau^4, 1) : C^n \to \mathscr{E}_3$ and since the morphisms $(\lambda^2, \alpha^2\tau^3, 1)$ and $(\lambda^4, \alpha^4\tau^4, 1)$ have the same right end, it follows from Proposition 6.3.2 they are homotopy equivalent; likewise $(\lambda^4, \alpha^4\tau^5, 1)$ and $(\lambda^6, \alpha^6\tau^6, 1) : C^n \to \mathscr{E}_5$ are homotopic and so forth and finally $(\lambda^{k-1}, \alpha^{k-2}\tau^{k-1}, 1)$ and $(\nu, \alpha^k\tau^k, 1)$ are also homotopy equivalent. Recall that the homotopy is a family of maps $h : C^n_n \to A_{n+1}$ and so the left end map $\mathcal{J}_n \to \mathcal{A}$ is extensible over C_{n-1}. The sum $-\nu + \mu$ can be expressed as

$$-\nu + \mu = \mu - \lambda^2 + \lambda^2 - \lambda^4 + \cdots - \lambda^{k-2} + \lambda^{k-2} - \nu$$

hence it can extended over C_{n-1} as required.

For the converse, let the map $\mathcal{J}_n \to \mathcal{A}$ be extensible over C_{n-1} to a map $C_{n-1} \to \mathcal{A}$.

Then the ordered crossed n-fold extensions$(-\nu + \mu)C^n$ splits and hence μC^n and νC^n are similar. $\square$

We have a clear description of the trivial similarity class. It is the class containing $\mathbf{0}$. Thus the map $\mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A}) \xrightarrow{\sigma} \mathscr{M}(Q, \mathcal{A})$ is injective.

Using the fact that $(\mathbf{OGpd}_G(C_{n-1}, \mathcal{A}) = \mathbf{OGpd}_Q(C_{n-1}, \mathcal{A})$, it follows from Proposition 6.4.1 that $H^{n+1}(Q^I, \mathcal{A}^0)$ is the cokernel of the restriction $\mathbf{OGpd}_G(C_{n-1}, \mathcal{A}) \to \mathsf{OGPD}_Q(\mathcal{J}_n, \mathcal{A})$ for $n \geqslant 3$. By Lemma 6.4.3 the epimorphism $\mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A}) \xrightarrow{\sigma} \mathscr{M}(Q, \mathcal{A})$ induces an epimorphism $\Phi : H^{n+1}(Q^I, \mathcal{A}^0) \to \mathscr{M}(Q, \mathcal{A})$ defined by $[\nu] \mapsto [\nu C^n]$.

Theorem 6.4.5 (Main result)**.** *The map* $\Phi : H^{n+1}(Q^I, \mathcal{A}^0) \to \mathscr{M}(Q, A)$ *is an isomorphism of abelian groups*

Proof. The set $\mathscr{M}(Q, A)$ is an abelian group with respect to the Baer sum. We note that $H^{n+1}(Q^I, \mathcal{A}^0)$ is the cokernel of the restriction $\mathbf{OGpd}_G(C_{n-1}, \mathcal{A}) \to \mathbf{OGpd}_Q(\mathcal{J}_n, \mathcal{A})$ from Proposition 6.4.1 and hence an induced epimorphism $\Phi : H^{n+1}(Q^I, \mathcal{A}^0) \to \mathscr{M}(Q, \mathcal{A})$ following Lemma 6.4.3. By Lemma 6.4.4, σ is injective hence Φ is injective. Therefore Φ is an isomorphism of abelian groups as desired. $\square$

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